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A Coordinate-Transformed Arnoldi Algorithm for Generating Guaranteed Stable Reduced-Order Models of Arbitrary RLC Circuits.

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Abstract

Since the first papers on asymptotic waveform evaluation (AWE), Padé-based reduced order models have become standard for improving coupled circuit-interconnect simulation efficiency. Such models can be accurately computed using bi-orthogonalization algorithms like Padé via Lanczos (PVL), but the resulting Padé approximates can still be unstable even when generated from stable RLC circuits. For certain classes of RC circuits it has been shown that congruence transforms, like the Arnoldi algorithm, can generate guaranteed stable and passive reduced-order models. In this paper we present a computationally efficient model-order reduction technique, the coordinate-transformed Arnoldi algorithm, and show that this method generates arbitrarily accurate and guaranteed stable reduced-order models for arbitrary RLC circuits. Examples are presented which demonstrate the enhanced stability and efficiency of the new method.

1 Introduction

The dense three-dimensional packaging used in compact electronic systems often produce magnetic interactions which interfere with system performance. Such effects are difficult to simulate because they occur only as a result of an interaction between the field distribution in a complicated geometry of conductors, and the circuitry connected to those conductors. For structures small compared to a wavelength, electromagnetic interactions between conductors can be represented arbitrarily accurately using a densely coupled resistor, inductor, and capacitor (RLC) network [1]. Although it is possible to simulate coupled circuit-interconnect problems by including this densely coupled RLC network with the transistor models in a circuit simulator, this can be a very inefficient approach.

A standard way to improve the efficiency of coupled circuit-interconnect simulation is to use Padé-based reduced order models [2, 3, 4, 5, 6]. Accurate computation of such models can be accomplished using bi-orthogonalization algorithms like Padé via Lanczos (PVL) [7], but the resulting Padé approximates can still be unstable even when generated from stable RLC circuits. It has been shown that, for certain classes of RC circuits, congruence transforms, like the Arnoldi algorithm, can generate guaranteed stable and passive reduced-order models [8]. In this paper we present a computationally efficient model-order reduction technique, the coordinate-transformed Arnoldi algorithm, and use a congruence argument similar to the one in [8] to show that our method generates arbitrarily accurate and guaranteed stable reduced-order models for general RLC circuits.

In the next section we briefly describe background on RLC circuit formulation, model-order reduction, Padé approximation, and Arnoldi methods. Then in section 3, we present a guaranteed stability theory comprising two steps: a coordinate transformation requiring the computation of a matrix square root and an Arnoldi iteration. In section 4, we show that the matrix square-root coordinate-transform can be performed implicitly as part of a coordinate-transformed Arnoldi algorithm, and that therefore it is *not* necessary to compute the matrix square-root. The results presented in section 5 include several examples. A simple RC circuit is examined to show that the generated Padé approximate is unstable but the coordinate-transformed Arnoldi algorithm produces a stable reduced-order model. Then, results are presented comparing the accuracy of the model-order reduction methods on an RLC filter, a low-noise amplifier and an equivalent circuit for a three-dimensional electromagnetic problem modeled via PEEC [1]. Fi-

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nally, in section 6, we present conclusions and acknowledgments.

2 Background

2.1 Formulation

If the modified nodal analysis approach is used to generate a system of equations for a network consisting of coupled inductors, capacitors, and resistors, the resulting n-node system has the form

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{i}} \end{bmatrix} = - \begin{bmatrix} \boldsymbol{G} & \boldsymbol{B} \\ -\boldsymbol{B}^T & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{i} \end{bmatrix} + \begin{bmatrix} \boldsymbol{i}_s \\ \mathbf{o} \end{bmatrix} \tag{1}$$

where $\boldsymbol{v} \in \mathbb{R}^N$ is the vector of N node voltages, $\boldsymbol{i} \in \mathbb{R}^M$ is the vector of M inductor currents, $\boldsymbol{i}_s \in \mathbb{R}^N$ is the vector of source currents, $\boldsymbol{C}, \boldsymbol{G} \in \mathbb{R}^{N\times N}$ are the symmetric nodal capacitance and conductor matrices respectively, $\boldsymbol{L} \in \mathbb{R}^{M\times M}$ is the symmetric branch inductance matrix, and $\boldsymbol{B} \in \mathbb{R}^{N\times M}$ is the incidence matrix associated with the inductor currents.

For the SISO (single transfer impedance) case, we can simplify the above system using the notation

$$\begin{aligned} \mathcal{L}\,\dot{\boldsymbol{x}} &= -\mathcal{R}\,\boldsymbol{x} + \boldsymbol{e}_j\,u \\ y &= {\boldsymbol{e}_k}^T \boldsymbol{x}. \end{aligned} \tag{2}$$

Here, $\boldsymbol{e}_j, \boldsymbol{e}_k, \in \mathbb{R}^{N+M}$ are the j-th and k-th unit vectors associated with computing the transfer impedance Z_{jk} from the j-th branch to the k-th node, and

$$\boldsymbol{x} \equiv \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{i} \end{bmatrix} \quad \mathcal{L} \equiv \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L} \end{bmatrix} \quad \mathcal{R} \equiv \begin{bmatrix} \boldsymbol{G} & \boldsymbol{B} \\ -\boldsymbol{B}^T & \boldsymbol{O} \end{bmatrix}. \tag{3}$$

Below we will use the more general notation

$$\begin{aligned} \boldsymbol{A}\,\dot{\boldsymbol{x}} &= \boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^T \boldsymbol{x}. \end{aligned} \tag{4}$$

where in our case $\boldsymbol{A} = -\mathcal{R}^{-1}\mathcal{L} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} = -\mathcal{R}^{-1}\boldsymbol{e}_j$, $\boldsymbol{c} = \boldsymbol{e}_k$ and $n = N + M$.

From (4), the transfer impedance is given by

$$Z_{jk}(s) = \frac{y(s)}{u(s)} = -\boldsymbol{c}^T \left(\boldsymbol{I} - s\boldsymbol{A}\right)^{-1} \boldsymbol{b} \tag{5}$$

where s is the Laplace transform variable.

A reduced-order model for (4) is the SISO system

$$\begin{aligned} \boldsymbol{A}_q\,\dot{\boldsymbol{x}}_q &= \boldsymbol{x}_q + \boldsymbol{b}_q u \\ \tilde{y} &= \boldsymbol{c}_q^T \boldsymbol{x}_q \end{aligned} \tag{6}$$

where $\boldsymbol{x}_q, \boldsymbol{b}_q, \boldsymbol{c}_q \in \mathbb{R}^q$, $\boldsymbol{A}_q \in \mathbb{R}^{q\times q}$ and q is presumably much smaller than n. The model-order reduction problem is then finding the smallest $\boldsymbol{A}_q$, $\boldsymbol{b}_q$ and $\boldsymbol{c}_q$ such that

$$\tilde{Z}_{jk}(s) = \frac{\tilde{y}(s)}{u(s)} = -{\boldsymbol{c}_q}^T \left(\boldsymbol{I} - s\boldsymbol{A}_q\right)^{-1} \boldsymbol{b}_q \tag{7}$$

approximates $Z_{jk} = \frac{y(s)}{u(s)}$ with sufficient accuracy.

2.2 Padé Approximations

The reason for the popularity of Padé approximates in circuit simulation is that it provides a systematic method for enforcing successively more accurate representations of the approach to steady-state. More formally, if the transfer impedance Z_{jk} (5) is expanded in a McLaurin series,

$$\boldsymbol{Z}_{jk}(s) = -\boldsymbol{c}^T \left(\boldsymbol{I} - s\boldsymbol{A}\right)^{-1} \boldsymbol{b} = -\sum_{k=0}^{\infty} m_k s^k. \tag{8}$$

where

$$m_k = \boldsymbol{c}^T \boldsymbol{A}^k \boldsymbol{b} \tag{9}$$

is known as the k^{th} moment of the transfer function, then a (diagonal) Padé approximation of q^{th} order is defined as the rational function

$$G_q^P(s) = \frac{b_{q-1}s^{q-1} + \cdots + b_1 s + b_0}{a_q s^q + a_{q-1}s^{q-1} + \cdots + a_1 s + 1} \tag{10}$$

whose coefficients are selected to match the first $2q$ moments of the transfer function (5).

Low order Padé approximates can be computed using direct evaluation of the moments, followed by a moment-matching procedure [9, 2]. In order to accurately compute higher order Padé approximates, it is necessary to use successive bi-orthogonalization combined with lookahead, as in the more recent nonsymmetric Lanczos algorithms [10, 11]. Although nonsymmetric Lanczos methods plus lookahead can be used to generate Padé approximates of arbitrarily high order, there is *no guarantee that a given approximate will be stable.* It is therefore essential to postprocess the Padé approximate before using it in a circuit simulation program.

2.3 Arnoldi-based Model Order Reduction

Padé approximates are in one sense optimal; they match as many moments as there are free coefficients in the reduced-order transfer function. It is possible to trade some of this optimality to gain guaranteed stability, at least for the case of RLC circuits with positive elements, using a model-order reduction algorithm based on the Arnoldi process. The Arnoldi approach is similar to Lanczos-style algorithms in that it creates an orthonormal basis for the Krylov subspace $\mathcal{K}_k(\boldsymbol{A}, \boldsymbol{b}) = span\{\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}, \cdots, \boldsymbol{A}^{k-1}\boldsymbol{b}\}$. And just like the Lanczos process, the Arnoldi algorithm is a better conditioned process than direct evaluation of the moments because it generates an orthogonal set of vectors which span $\boldsymbol{A}^k\boldsymbol{b}, k = 0, \ldots, 2q-1$.

After q steps, the Arnoldi algorithm returns a set of q orthonormal vectors, as the columns of the matrix $\boldsymbol{V}_q \in \mathbb{R}^{n\times q}$, and a $q \times q$ upper Hessenberg (tridiagonal plus upper triangular) matrix $\boldsymbol{H}_q$ whose entries are the scalars $h_{i,j}$ generated by the Arnoldi algorithm. These two matrices satisfy the following relationship:

$$\boldsymbol{A}\ \boldsymbol{V}_q = \boldsymbol{V}_q\ \boldsymbol{H}_q + h_{q+1,q}\ \boldsymbol{v}_{q+1}\ \boldsymbol{e}_q^T \qquad (11)$$

where $\boldsymbol{e}_q$ is the q^{th} unit vector in $\mathbb{R}^q$. From (11), it can easily be seen that after q steps of an Arnoldi process, for $k < q$,

$$\boldsymbol{A}^k\ \boldsymbol{b} = ||\boldsymbol{b}||_2\ \boldsymbol{A}^k\ \boldsymbol{V}_q\ e_1 = ||\boldsymbol{b}||_2\ \boldsymbol{V}_q\ \boldsymbol{H}_q^k\ e_1. \qquad (12)$$

With this relation, the moments (9) can be related to $\boldsymbol{H}_q$ by

$$m_k = \boldsymbol{c}^T\ \boldsymbol{A}^k\ \boldsymbol{b} = \underbrace{||\boldsymbol{b}||_2\ \boldsymbol{c}^T\ \boldsymbol{V}_q}_{\boldsymbol{c}_q^T}\ \underbrace{\boldsymbol{H}_q^k}_{\boldsymbol{A}_q^k}\ \underbrace{\boldsymbol{e}_1}_{\boldsymbol{b}_q} \qquad (13)$$

and so, by analogy with (9), the q^{th} order Arnoldi-based approximation to $\boldsymbol{Z}_{ij}$ can be written as

$$\boldsymbol{G}_q^A(s) = ||\boldsymbol{b}||_2\ \boldsymbol{c}^T\ \boldsymbol{V}_q\ (\boldsymbol{I} - s\boldsymbol{H}_q)^{-1}\ e_1 \qquad (14)$$

corresponding to the state-space realization using the triplet $\boldsymbol{\mathcal{S}}_q^A = [\boldsymbol{A}_q, \boldsymbol{b}_q, \boldsymbol{c}_q]$ where $\boldsymbol{A}_q = \boldsymbol{H}_q$, $\boldsymbol{b}_q = \boldsymbol{e}_1$, and $\boldsymbol{c}_q = ||\boldsymbol{b}||_2\ \boldsymbol{V}_q^T\ \boldsymbol{c}$.

3 Guaranteed Stability Theorems

In this section we extend the basic theoretical strategy in [8], which was applied to resistor-capacitor networks, to yield a result which guarantees the stability of the Arnoldi-generated reduced-order models for arbitrary RLC circuits. The stability result given below requires that the Arnoldi algorithm be applied to a coordinate-transformed version of (1) using the square-root of the $\boldsymbol{L}$ and $\boldsymbol{C}$ matrices. In the next section we will show that the coordinate-transformation can be efficiently "folded" into the Arnoldi algorithm and that no matrix square-roots need be computed. We first give some basic lemmas, then prove the main theorem in a general setting, and finally we show that the theorem applies to systems generated from RLC circuits.

3.1 Definitions and Basic Lemmas.

Throughout this section it is assumed that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and the Arnoldi process has been used to construct an Hessenberg matrix $\boldsymbol{H}_q \in \mathbb{R}^{q\times q}$ such that

$$\boldsymbol{V}_q^T \boldsymbol{A} \boldsymbol{V}_q = \boldsymbol{H}_q, \qquad (15)$$

where the matrix $\boldsymbol{V}_q \in \mathbb{R}^{n\times q}$, generated by the Arnoldi process, has q orthonormal columns.

We will use the following definitions:

Definition 1 *The real matrix $\boldsymbol{A}$ is said to be negative semidefinite if*

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \leq 0$$

for any non-zero vector $\boldsymbol{x}$.

Note that our definition does not make the typical assumption [12] that $\boldsymbol{A}$ is symmetric.

Definition 2 *The real matrix $\boldsymbol{A}$ is said to be (strictly) stable if all its eigenvalues have (negative) nonpositive real parts.*

Since we have not assumed symmetry in Definition 1, the next lemma is not entirely obvious.

Lemma 3 *If the real matrix $\boldsymbol{A}$ is negative semidefinite then: $\boldsymbol{A}$ is stable; $\boldsymbol{A}^{-1}$ is negative semidefinite; and if $\boldsymbol{B}$ is any symmetric matrix, $\boldsymbol{BAB}$ is negative semidefinite.*

For a proof of the above lemma, see [13].

3.2 Main Result

Using the above definitions and lemmas,

Theorem 4 *If the real matrix $\boldsymbol{A}$ is negative semidefinite then the matrix $\boldsymbol{H}_q$ generated by the Arnoldi process is stable.*

Proof. Let $\boldsymbol{x}$ be an arbitrary non-zero vector in $\mathbb{R}^q$. Then we have

$$\boldsymbol{x}^T \boldsymbol{H}_q \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{V}_q^T \boldsymbol{A} \boldsymbol{V}_q \boldsymbol{x} = (\boldsymbol{V}_q \boldsymbol{x})^T \boldsymbol{A} (\boldsymbol{V}_q \boldsymbol{x}) \leq 0,$$

where the first equality results from the definition of $\boldsymbol{H}_q$ (see Equation 15) and the inequality results from the fact that $\boldsymbol{A}$ is assumed negative semidefinite. Lemma 3 allows us to conclude that since $\boldsymbol{H}_q$ is negative semidefinite, it is stable. $\square$

Given the result in Theorem (4), we can insure that the Arnoldi algorithm will produce a stable reduced-order model if the associated system matrix $\boldsymbol{A}$ is negative semidefinite. Although the matrices $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{R}}$, generated by modified nodal analysis of an RLC circuit with positive elements, are in general positive semidefinite, the matrix $\boldsymbol{A} = -\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}$ is not necessarily negative semidefinite. It is well known, however, that the property of negative or positive definiteness of a matrix depends on the basis chosen for the state space $\mathbb{R}^n$. A natural question then is whether there is a change of coordinates in the state space such that the resulting system matrix is negative definite.

The answer to the above question is indeed affirmative. Consider the change of variable

$$\tilde{\boldsymbol{x}} = \boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{x} \tag{16}$$

where $\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$ is the unique symmetric, positive semidefinite square root of the symmetric, positive semidefinite matrix $\boldsymbol{\mathcal{L}}$. From this it follows that (2) can be written as

$$-\left(\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\right)\dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{x}} - \boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{e}_j u \tag{17}$$

and that the output equation becomes

$$y = \boldsymbol{e}_k^T\boldsymbol{\mathcal{L}}^{-\frac{1}{2}}\tilde{\boldsymbol{x}}. \tag{18}$$

The modified system matrix is now given by

$$\tilde{\boldsymbol{A}} = -\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}^{\frac{1}{2}} \tag{19}$$

while the input and output vectors are given by

$$\tilde{\boldsymbol{b}} = -\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{b} = -\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{e}_j \qquad \tilde{\boldsymbol{c}}^T = \boldsymbol{c}^T\boldsymbol{\mathcal{L}}^{-\frac{1}{2}} = \boldsymbol{e}_k^T\boldsymbol{\mathcal{L}}^{-\frac{1}{2}}. \tag{20}$$

As can be easily verified, moments are invariant under a change of coordinates in the state space. Therefore, a reduced order model that matches the moments of (19) and (20) will also match the moments of the original system.

The coordinate change leads us to the main circuit-specific result.

Theorem 5 *If the matrix $-\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$ is generated from modified nodal analysis of an RLC circuit with positive elements, then the $\boldsymbol{H}_q$ generated by the Arnoldi process applied to $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ is stable.*

Proof. As a consequence of the result of Theorem 4, it is only necessary to show that $-\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$ is negative semidefinite. To prove this result, we begin by demonstrating that $-\boldsymbol{\mathcal{R}}$ is negative semidefinite. Using $\boldsymbol{\mathcal{R}}$'s definition in (3),

$$\boldsymbol{x}^T\boldsymbol{\mathcal{R}}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{i} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{G} & \boldsymbol{B} \\ -\boldsymbol{B}^T & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{i} \end{bmatrix}. \tag{21}$$

Carrying out the matrix multiplication reveals

$$-\boldsymbol{x}^T\boldsymbol{\mathcal{R}}\boldsymbol{x} = -\boldsymbol{v}^T\boldsymbol{G}\boldsymbol{v} \leq 0 \tag{22}$$

because the $\boldsymbol{G}$ matrix is positive definite, or more intuitively, the power dissipated by a network of positive resistors is always positive.

Combining (22) with Lemma 3 implies that $-\boldsymbol{\mathcal{R}}^{-1}$ is negative semidefinite. It then also follows from Lemma 3 that $-\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$ is negative semidefinite, proving the theorem. □

Theorem 5 holds only for a reduced-order matrix $\boldsymbol{H}_q$ obtained using the Arnoldi procedure. We will show in Section 5 that the Lanczos algorithm can indeed produce an unstable reduced-order model even for a circuit which generates a symmetric negative definite matrix.

4 Coordinate-Transformed Arnoldi Algorithm

In order to obtain the stable transfer function corresponding to the system in (19) and (20), the Arnoldi algorithm must be applied to the matrix $-\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$ and the input vector $-\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{\mathcal{R}}^{-1}\boldsymbol{e}_j$. This might lead to the belief that the computation of $\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$, potentially a costly operation, is needed. Such computation can be altogether avoided by using a modified Arnoldi algorithm which generates the $\boldsymbol{H}_q$ associated with the transformed system matrix and input vector, but does not require explicit computation of $\boldsymbol{\mathcal{L}}^{\frac{1}{2}}$.

This modified Arnoldi algorithm uses a "hiding the square-root" trick commonly used when preconditioning Conjugate-Gradient schemes [14]. The key idea is that most of the operations involve inner products of the form

$$\left(\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{u}\right)^T\boldsymbol{\mathcal{L}}^{\frac{1}{2}}\boldsymbol{y}. \tag{23}$$

If $\boldsymbol{\mathcal{L}}$ is symmetric, which is the case for RLC circuits, then (23) can be rewritten as $\boldsymbol{u}^T\boldsymbol{\mathcal{L}}\boldsymbol{y}$, which no longer requires the square root. The presence of the matrix $\boldsymbol{\mathcal{L}}$ can be construed as endowing $\mathbb{R}^n$ with an induced dot product, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{L}}} = \boldsymbol{y}^T\boldsymbol{\mathcal{L}}\boldsymbol{x}$, thus leading to what we term a modified $\boldsymbol{\mathcal{L}}$-orthogonal version of the Arnoldi

Algorithm. It can be shown that the state-space representation and the transfer-function of the reduced-order model can be entirely determined from the outputs of this modified $\mathcal{L}$-orthogonal Arnoldi algorithm on the Krylov subspace $\mathcal{K}_q(-\mathcal{R}^{-1}\mathcal{L}, -\mathcal{R}^{-1}\boldsymbol{b})$, whose matrix and input vector correspond to those of the original system in Eqn. (2). Furthermore, the algorithm can be adapted to have as inputs the matrices $\mathcal{L}$ and $\mathcal{R}$, thus avoiding explicit computation of $\mathcal{R}^{-1}$. For details see [13]. This algorithm is shown as Algorithm 1.

Algorithm 1 **(Modified $\mathcal{L}$-orthogonal Arnoldi)**

arnoldi(input $\mathcal{L}, \mathcal{R}, \boldsymbol{b}, q$; output $\boldsymbol{U}_q, \boldsymbol{v}_{q+1}, \boldsymbol{H}_q, h_{q+1,q}$)

{

  Initialize:

    Solve : $\mathcal{R}\boldsymbol{u}_0 = -\boldsymbol{b}$

    $\boldsymbol{z}_0 = \mathcal{L}\boldsymbol{u}_0$

    $h_{0,0} = \sqrt{\boldsymbol{u}_0^T \boldsymbol{z}_0}$

    $\boldsymbol{z}_1 = \boldsymbol{z}_0 / h_{0,0}$

    $\boldsymbol{u}_1 = \boldsymbol{u}_0 / h_{0,0}$

  for $(j = 1;\ j <= q;\ j++)$ {

    Solve $\mathcal{R}\boldsymbol{w} = \boldsymbol{z}_j$

    for $(i = 1;\ i <= j-1;\ i++)$ {

      $h_{i,j} = \boldsymbol{w}^T \boldsymbol{z}_i$

      $\boldsymbol{w} = \boldsymbol{w} - h_{i,j}\boldsymbol{u}_i$

    }

    $\boldsymbol{z}_{j+1} = \mathcal{L}\boldsymbol{w}$

    $h_{j+1,j} = \sqrt{\boldsymbol{w}^T \boldsymbol{z}_{j+1}}$

    if $(h_{j+1,j} \neq 0)$ {

      $\boldsymbol{z}_{j+1} = \boldsymbol{z}_{j+1} / h_{j+1,j}$

      $\boldsymbol{u}_{j+1} = \boldsymbol{w} / h_{j+1,j}$

    }

  }

  $\boldsymbol{U}_q = [\boldsymbol{u}_1 \cdots \boldsymbol{u}_q]$

  $\boldsymbol{H}_q = (h_{i,j}),\quad i, j = 1, \cdots, q$

In Algorithm 1, it is only necessary to be able to multiply $\mathcal{L}$ by a vector, and then solve a system with the matrix $\mathcal{R}$. For general problems this implies that the $\mathcal{R}$ must be factored, typically using sparse matrix techniques. For interconnect problems with a near-tree like structure, faster algorithms have been presented [15].

The computational cost of Algorithm 1 is that of executing one sparse LU factorization for $\mathcal{R}$, $q+1$ matrix-vector products for computing the $\boldsymbol{z}_j$ vectors, and $q+1$ back substitutions for computing $\boldsymbol{u}_0$ and the $\boldsymbol{w}$ vectors. It has therefore about the same computational cost as PVL, one back substitution being roughly equivalent to one matrix-vector product.

Finally, note that $\boldsymbol{H}_q$ has a special structure if both the $\mathcal{L}$ and $\mathcal{R}$ matrices are symmetric. This would be the case for either RL or RC circuits, but *not* generally for RLC circuits. In this symmetric case, the output matrix $\boldsymbol{H}_q$ of the modified $\mathcal{L}$-orthogonal Arnoldi algorithm is tridiagonal with negative coefficients on the diagonal and positive coefficients on the subdiagonals. In addition, the back orthogonalization can be truncated to only two steps.

5 Experimental Results

In this section we present several examples. A simple RC circuit is examined to show that the generated Padé approximate is unstable but the coordinate-transformed Arnoldi algorithm produces a stable reduced-order model. Then, results are presented comparing the accuracy of the model-order reduction methods for a low-noise amplifier. Finally, results are shown for a lumped-equivalent circuit for a three dimensional electromagnetic problem modeled via PEEC.

5.1 Simple RC Example

Consider the RC circuit in Fig. 1. Assuming all the capacitors are one Farad, appropriately choosing the resistors, and using nodal analysis, the system matrix for model-order reduction is

$$\boldsymbol{A} = -\boldsymbol{R} = -\boldsymbol{G}^{-1} = \begin{bmatrix} 1 & r & r^2 & r^3 \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r^3 & r^2 & r & 1 \end{bmatrix}.$$

The matrix $-\boldsymbol{R}$ is symmetric and negative definite. However, the 3^{rd} order Padé approximate computed using the input vector $\boldsymbol{b} = [1\ \ r\ \ r^2\ \ r^3]^T$ and the output vector $\boldsymbol{c} = -[0\ \ -1\ \ -r\ \ -r^2]^T$ is unstable. This is shown in Table 1, which displays the poles obtained from the Padé approximate (computed using the PVL algorithm) and the Arnoldi algorithms (Here $r = 0.4907783849587564$). As is also clear from the table, the Arnoldi-based model is stable, which is guaranteed by Theorem 5. Furthermore, the Arnoldi model is also quite accurate. In fairness to the Padé approach using Lanczos, it is always possible to increase the order of the approximate and then postpro-

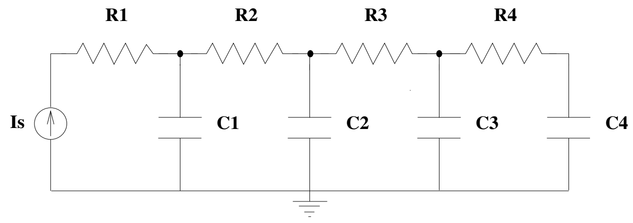

Figure 1: RC-circuit that shows that the Lanczos algorithm can produce an unstable model even if the system is described by a symmetric positive definite matrix.

Padé poles	Arnoldi poles	Exact poles
-2.0593187e+00	-2.0593862e+00	-2.0594789e+00
-4.9929056e-01	-1.0021690e+00	-1.0072092e+00
4.9116674e-01	-5.0557753e-01	-5.4951008e-01
—	—	-3.8380185e-01

Table 1: Comparison of poles obtained from Padé and Arnoldi reduced-order models of 3^{rd} order to the exact poles of the system resulting from the circuit in Fig 1.

cess the reduced-order model to eliminate the unstable modes.

5.2 Low-Noise Amplifier Example

This example illustrates the relative accuracies of the Padé, and the coordinate-transformed Arnoldi algorithms. It also gives results for the block generalization of the Arnoldi algorithm, though the algorithm is not described above. The circuit used for this example is a low-noise amplifier designed for radio-frequency applications. The circuit and its extracted netlist were introduced in [16] and we applied our algorithms to the matrices that describe the resulting linear circuit. The amplifier is a two-port and is therefore modeled as a two-input, two-output system.

The 2×2 matrix transfer-function that fully characterizes the circuit was approximated using both the Padé-via-Lanczos algorithm, the Arnoldi algorithm and the block Arnoldi algorithm. Figures 2 plots the magnitude of the amplifier gain. As is clear from the frequency response plots, the Arnoldi and Padé approximations are of similar accuracy.

5.3 PEEC Example

The following example was introduced in [7]. The network is the lumped-element equivalent circuit for a three-dimensional problem modeled via PEEC. The circuit consists of 2100 capacitors, 172 inductors and

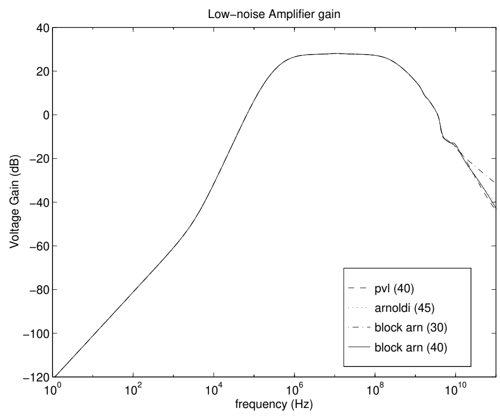

Figure 2: Low-noise amplifier voltage gain approximations. Shown in the figure are the following approximants: 40^{th} order Padé, 45^{th} order Arnoldi, 30^{th} and 40^{th} order block Arnoldi. The 40^{th} order block Arnoldi results are indistinguishable from the exact gain.

6990 inductive couplings, resulting in a 304×304 dense MNA matrix. In [7] it was shown that a $60^{\underline{th}}$ order approximation computed with PVL was able to reproduce the exact transfer function of the equivalent circuit. However, it was also reported, that some of the poles obtained with the PVL algorithm had positive real parts, albeit small. If the approximation is intended to be used within a circuit simulator, post-processing is required to eliminate such poles, which can be done if their residues are very small. The approximation obtained with the modified $\mathcal{L}$-orthogonal Arnoldi algorithm, shown for comparison in Figure 3, can be seen to be of comparable accuracy, and is guaranteed stable. In fact the converged poles in this approximation all have nonpositive real parts. It can therefore be used in a circuit simulator unmodified.

6 Conclusions

In this paper we presented a solution of the stability problem of reduced-order models within the paradigm of model-order reduction by moment matching. Our solution is a two-step process, including a state-space transformation step and an Arnoldi iteration step applied to the transformed state-space matrix and input vector. Our solution, which provably guarantees the stability of reduced-order model, is general in that it applies to RLC circuits and computationally elegant in that the two steps can be seamlessly combined in one single algorithm that does not require the explicit computation of the state-space transformation. In the

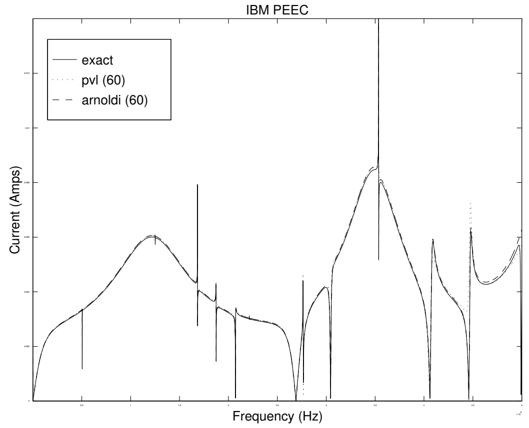

Figure 3: Circuit for 3-D problem modeled via PEEC. Shown in the figure are the exact solution, and the $60^{\underline{th}}$ order PVL and Modified Arnoldi approximations. Both are able to reproduce the transfer function with high accuracy.

special cases of RC or RL problems, the coordinate-transformed Arnoldi algorithm produces a symmetric tridiagonal reduced-order system matrix. We have exhibited a simple example which shows that simple RC circuits can lead to Padé approximates that are unstable but for which the coordinate-transformed Arnoldi algorithm is stable. The numerical examples that we have presented include a low-noise amplifier and a large RLC PEEC model, both of which could be modeled with reduced-order models that have the merit of being not only accurate but also stable at any reduction order.

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References

[1] Albert E. Ruehli. Equivalent Circuit Models for Three-Dimensional Multiconductor Systems. *IEEE Transactions on Microwave Theory and Techniques*, MTT-22(3):216–221, March 1974.

[2] Lawrence T. Pillage and Ronald A. Rohrer. Asymptotic Waveform Evaluation for Timing Analysis. *IEEE Trans. CAD*, 9(4):352–366, April 1990.

[3] Eli Chiprout and Michael Nakhla. Generalized Moment-Matching Methods for Transient Analysis of Interconnect Networks. In 29^{th} *ACM/IEEE Design Automation Conference*, pages 201–206, Anaheim, California, June 1992.

[4] Eli Chiprout, Hansruedi Heeb, Michael S. Nakhla, and Albert E. Ruehli. Simulating 3-D retarded interconnect models using complex frequency hopping (CFH). In *International Conference on Computer Aided-Design*, pages 66–72, Santa Clara, California, November 1993.

[5] K. Gallivan, E. Grimme, and P. Van Dooren. Asymptotic Waveform Evaluation via a Lanczos Method. *Applied Mathematics Letters*, 7(5):75–80, 1994.

[6] J. E. Bracken, V. Raghavan, and R. A. Rohrer. Interconnect Simulation with Asymptotic Waveform Evaluation. *IEEE Trans. Circuits Syst.*, 39(11):869–878, November 1992.

[7] P. Feldmann and R. W. Freund. Efficient linear circuit analysis by Pad´e approximation via the Lanczos process. *IEEE Iransactions on Computer-Aided Design of Integrated Circuits and Systems*, 14:639–649, 1995.

[8] K. J. Kerns, I. L. Wemple, and A. T. Yang. Stable and efficient reduction of substrate model networks using congruence transforms. In *IEEE/ACM International Conference on Computer Aided Design*, pages 207 – 214, San Jose, CA, November 1995.

[9] F. P. Lees. The determination of the Moments of the Impulse Response of Chemical Processes from the basic transformed equations. *Chemical Engineering Science*, 24:1607–1613, 1969.

[10] K. Gallivan, E. Grimme, and P. Van Dooren. Multipoint padé approximants of large-scale systems via a two-sided rational krylov algorithm. In 33^{rd} *IEEE Conference on Decision and Control*, Lake Buena Vista, FL, December 1994.

[11] Peter Feldmann and Roland W. Freund. Efficient linear circuit analysis by Padé approximation via the Lanczos process. In *EURO-DAC'94 with EURO-VHDL'94*, September 1994.

[12] Gilbert Strang. *Linear Algebra and its Applications*. Academic Press, New York, NY, second edition, 1980.

[13] L. M. Silveira, M. Kamon, I. M. Elfadel, and J. White. Coupled circuit-interconnect analysis using Arnoldi-based model order reduction. *IEEE Trans. on CAD*, 1995. In preparation.

[14] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. The John Hopkins University Press, Baltimore, Maryland, 1983.

[15] Curtis L. Ratzlaff and Lawrence T. Pillage. RICE: Rapid interconnect circuit evaluation using AWE. *IEEE Trans. CAD*, 13(6):763–776, June 1994.

[16] Peter Feldmann and Roland W. Freund. Reduced-order modeling of large linear subcircuits via a block lanczos algorithm. In 32^{nd} *ACM/IEEE Design Automation Conference*, pages 474–479, San Francisco, California, June 1995.