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STABILITY CRITERIA FOR ARNOLDI-BASED MODEL-ORDER REDUCTION

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ABSTRACT

Padé approximation is an often-used method for reducing the order of a finite-dimensional, linear, time invariant, signal model. It is known to suffer from two problems: numerical instability during the computation of the Padé coefficients and lack of guaranteed stability for the resulting reduced model *even* when the original system is stable. In this paper, we show how the numerical instability problem can be avoided using the Arnoldi algorithm applied to an appropriately chosen Krylov subspace. Moreover, we give an easily computable sufficient condition on the system matrix that guarantees the stability of the reduced model at any approximation order.

1. INTRODUCTION

Consider the linear, time-invariant (LTI) system

$$\begin{aligned} \boldsymbol{x}(t+1) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \\ y(t) &= \boldsymbol{c}\boldsymbol{x}(t) \end{aligned} \qquad (1)$$

where $\boldsymbol{x}(t), \boldsymbol{b}, \boldsymbol{c}^T \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}$, and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. We will assume this system both reachable and observable, which means that the ranks of the reachability matrix $(\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b})$ and observability matrix $\left(\boldsymbol{c}^T, \boldsymbol{c}^T\boldsymbol{A}^T, \ldots, \boldsymbol{c}^T(\boldsymbol{A}^{n-1})^T\right)$ are both equal to n. The transfer function of this linear system is[1] $G(z) = z\boldsymbol{c}(\boldsymbol{I}_n - z\boldsymbol{A})^{-1}\boldsymbol{b}$, where $\boldsymbol{I}_p$ is the identity matrix of order p. The rational function $G(z)$ can also be written as the power series

$$G(z) = \sum_{k=0}^{\infty} \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{b}z^{k+1}. \qquad (2)$$

The scalars $g_k \triangleq \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{b}$ are called the Markov coefficients of (1). They are in fact its impulse response. If the sequence g_k is summable, $G(z)$ is said to be stable. We denote by λ_i the i-th eigenvalue of $\boldsymbol{A}$ and define the spectral radius of $\boldsymbol{A}$ as $\rho(\boldsymbol{A}) = \max\{|\lambda_i|, 1 \le i \le n\}$. The stability of the the LTI system (1) is then equivalent to $\rho(\boldsymbol{A}) < 1$. In this paper, we will speak interchangeably of the stability of the transfer function or the stability of its system matrix $\boldsymbol{A}$.

[1] The delay operator is denoted by z rather than z^{-1}.

Assuming that the dimension of the state space n is very large and that we are given an integer $q < n$, the Padé approximation method aims at finding a triplet $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}) \in \mathbb{R}^{q\times q} \times \mathbb{R}^{q\times 1} \times \mathbb{R}^{1\times q}$, such that the transfer function $\tilde{G}(z) = z\tilde{\boldsymbol{c}}(\boldsymbol{I}_q - z\tilde{\boldsymbol{A}})^{-1}\tilde{\boldsymbol{b}}$ of the reduced system approximates the transfer function $G(z)$ in the sense that there is perfect matching between the coefficients of the Markov coefficients of the original and reduced models up to a certain order m, i.e.,

$$\tilde{g}_k = \tilde{\boldsymbol{c}}\tilde{\boldsymbol{A}}^k\tilde{\boldsymbol{b}} = \boldsymbol{c}\boldsymbol{A}^k\boldsymbol{b} = g_k, \quad 0 \le k \le m-1.$$

The classical procedure ([1], Chapter 3) to solve this problem is to solve a Hankel linear system in which the Hankel matrix is based on the Markov coefficients of (1) and the unknowns are the coefficients of the rational function $\tilde{G}(z)$. This procedure suffers from two main problems. The first is computational and is related to the computation of the Markov coefficients of the original system. This computation involves the power iterations of the large system matrix $\boldsymbol{A}$. It is well known ([2], Chapter 7), that for $\boldsymbol{r} \in \mathbb{R}^n$ the iterates $\boldsymbol{A}^k\boldsymbol{r}$ converges generically to the eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue of the largest magnitude. In other words, the Markov coefficients become close to each other, thus making the Hankel system very ill-conditioned [3].

Another more fundamental problem is that even when the LTI system (1) is stable and the computations are well-conditioned, there is no guarantee that the resulting reduced order system $\tilde{G}(s)$ will be stable. This instability could occur *even* when the original system is an FIR filter, i.e, $H(z) = \sum_{k=0}^{N} h_k z^k$ ([1], Example 3.9).

In this paper, we address these two problems and show that the Arnoldi algorithm (see [4] and

Algorithm 2.1 (Arnoldi Algorithm)

```
arnoldi(input A, b, q;
  output V_q, v_{q+1}, H_q, h_{q+1,q})
{  v_1 = b/||b||_2
   for (j = 1; j <= q; j++) {
      w = A v_j
      for (i = 1; i <= j-1; i++) {
         h_{i,j} = w^T v_i
         w = w - h_{i,j} v_i
      }
      h_{j+1,j} = ||w||_2
      if (h_{j+1,j} != 0) {
         v_{j+1} = w/h_{j+1,j}
      }
   }
   V_q = [v_1 ... v_q]
   H_q = (h_{i,j}),  i,j = 1,...,q
}
```

the references therein) provides a numerically stable way for obtaining a reduced-order model. We also show that when the system matrix $\boldsymbol{A}$ is symmetric and stable, the Arnoldi reduced-order matrix $\tilde{\boldsymbol{A}}$ is guaranteed stable at any approximation order q. When the $\boldsymbol{A}$ is not symmetric, we provide an easily computable sufficient condition for guaranteeing the stability of $\tilde{\boldsymbol{A}}$ at any approximation order q.

2. REDUCED-ORDER MODEL

The computation of the Markov coefficients involves the power iterates $\boldsymbol{A}^k\boldsymbol{b}$. It is therefore natural to consider the Krylov subspace $\mathcal{K}_q(\boldsymbol{A},\boldsymbol{b}) = span\{\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}, \cdots, \boldsymbol{A}^{q-1}\boldsymbol{b}\}$. Because of the reachability assumption, this subspace is of dimension q. The essence of the Arnoldi algorithm is to use the Gram-Schmidt procedure to build an orthonormal basis $\boldsymbol{V}_q = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_q\}$ of the Krylov subspace $\mathcal{K}_q(\boldsymbol{A},\boldsymbol{b})$. At the k-th step of the algorithm as shown in (2.1), a unit-norm vector $\boldsymbol{v}_k$ is constructed such that $\boldsymbol{v}_k$ is orthogonal to $\mathcal{K}_{k-1}(\boldsymbol{A},\boldsymbol{b})$.

After q steps, the Arnoldi algorithm returns a set of q orthonormal vectors, as the columns of the matrix $\boldsymbol{V}_q \in \mathbb{R}^{m\times q}$, and a $q\times q$ upper Hessenberg matrix $\boldsymbol{H}_q = [h_{i,j}]$. These two matrices satisfy the following relationship:

$$\boldsymbol{A}\boldsymbol{V}_q = \boldsymbol{V}_q\boldsymbol{H}_q + h_{q+1,q}\boldsymbol{v}_{q+1}\boldsymbol{e}_q^T \tag{3}$$

where $\boldsymbol{e}_q$ is the q-th unit vector in $\mathbb{R}^n$, and $\boldsymbol{v}_{q+1} \in \mathbb{R}^n$ is orthonormal to the columns of $\boldsymbol{V}_q$. Using the fact that $\boldsymbol{V}_q^T\boldsymbol{V} = \boldsymbol{I}_q$, we can write the above equation as

$$\boldsymbol{V}_q^T\boldsymbol{A}\boldsymbol{V}_q = \boldsymbol{H}_q. \tag{4}$$

Moreover, using the fact that $\boldsymbol{V}_q\boldsymbol{e}_1 = \boldsymbol{b}/||\boldsymbol{b}||_2$, it can be easily seen that after q steps of an Arnoldi process,

$$\boldsymbol{A}^k\boldsymbol{b} = ||\boldsymbol{b}||_2\boldsymbol{A}^k\boldsymbol{V}_q\boldsymbol{e}_1 = ||\boldsymbol{b}||_2\boldsymbol{V}_q\boldsymbol{H}_q^k\boldsymbol{e}_1, \quad 0\le k\le q-1, \tag{5}$$

which yields

$$\boldsymbol{c}\boldsymbol{A}^k\boldsymbol{b} = ||\boldsymbol{b}||_2\boldsymbol{c}\boldsymbol{V}_q\boldsymbol{H}_q^k\boldsymbol{e}_1, \qquad 0\le k\le q-1. \tag{6}$$

Therefore, if we choose for the reduced-order model the triplet $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}}) = (\boldsymbol{H}_q, \boldsymbol{e}_1, ||\boldsymbol{b}||_2\boldsymbol{c}\boldsymbol{V}_q)$, the original system and the reduced-order system will have q of their Markov coefficients matched. The transfer function of the reduced-order model is then

$$\tilde{G}(z) = z||\boldsymbol{b}||_2\boldsymbol{c}^T\boldsymbol{V}_q\left(\boldsymbol{I}_q - z\boldsymbol{H}_q\right)^{-1}\boldsymbol{e}_1 \tag{7}$$

Note that the state-space realization of the Arnoldi reduced-order model comes naturally in a *system Hessenberg form* [5]. This algorithm has been successfully used in [4] to produce reduced-order models for a variety of very large linear circuits encountered in the analysis and simulation of VLSI interconnect.

3. STABILITY

From now on we make the assumption that the system matrix $\boldsymbol{A}$ is stable, i.e., its spectral radius $\rho(\boldsymbol{A}) < 1$. We would like to find under what conditions the reduced matrix $\tilde{\boldsymbol{A}} = \boldsymbol{H}_q$ is itself stable. Formula (4) will be essential in answering this question.

Case 1: $\boldsymbol{A}$ is symmetric. The symmetric case occurs very often in practice especially in the context of scientific computing [3, 4]. When $\boldsymbol{A}$ is symmetric, (4) implies that the upper Hessenberg matrix $\boldsymbol{H}_q$ is symmetric and therefore tridiagonal. We denote the q *real* eigenvalues of $\boldsymbol{H}_q$ by $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \ldots \ge \tilde{\lambda}_q$. Because both $\boldsymbol{A}$ and $\boldsymbol{H}_q$ are symmetric, we can use the Rayleigh-Ritz quotients ([6], p. 176) to conclude that

$$\lambda_{\max} \geq \tilde{\lambda}_{\max} \geq \tilde{\lambda}_{\min} \geq \lambda_{\min}. \tag{8}$$

In other words, when $\boldsymbol{A}$ is symmetric, the spectral radius of $\tilde{\boldsymbol{A}}$ satisfies $\rho(\tilde{\boldsymbol{A}}) \leq \rho(\boldsymbol{A}) < 1$, i.e., the stability of the reduced-order model is guaranteed at any order whenever the original matrix $\boldsymbol{A}$ is stable.

In fact, using the theory of Paige-Kaniel ([2], Chapter 9), we can obtain much sharper results about the location of the eigenvalues of $\boldsymbol{H}_q$ with respect to those of $\boldsymbol{A}$.

Case 2: $\boldsymbol{A}$ is unsymmetric. When $\boldsymbol{A}$ is unsymmetric, it is not possible to use the Rayleigh-Ritz variational characterization of the eigenvalues. However, the concept of the *numerical* radius ([7], p. 7) defined as $r(\boldsymbol{A}) = \max\{|\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}|,\ \boldsymbol{x}^*\boldsymbol{x} = 1\}$ can be used to prove the following

Theorem 3.1 *Assume* $||A||_1 + ||A||_\infty < 2$. *Then* $\rho(\tilde{\boldsymbol{A}}) < 1$, *i.e., the reduced-order model is stable at any order.*

Proof. The proof uses the following three facts:

1. The spectral radius is always no larger than the numerical radius, i.e., $\rho(\boldsymbol{H}_q) \leq r(\boldsymbol{H}_q)$.
2. Orthogonal projections can not increase the numerical radius, i.e., $r(\boldsymbol{H}_q) \leq r(\boldsymbol{A})$.
3. The numerical radius of any matrix is always no greater than the average of its ℓ^1 and ℓ^∞ norms, i.e., $r(\boldsymbol{A}) \leq \frac{1}{2}(||\boldsymbol{A}||_1 + ||\boldsymbol{A}||_\infty)$.

The first two facts are not difficult to prove, while a proof of the third can be found in ([7], p. 31-33). Combining these facts leads to the conclusion that for any order $q \leq n$, $\rho(\tilde{\boldsymbol{A}}) = \rho(\boldsymbol{H}_q) < 1$, which means that the reduced-order model is stable at any order. □

The importance of the numerical radius stems from the fact that for *any* matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and *any* isometry $\boldsymbol{V} \in \mathbb{R}^{n\times q}, (\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}_q)$, we have $r(\boldsymbol{V}^T\boldsymbol{A}\boldsymbol{V}) \leq r(\boldsymbol{A})$. This latter inequality is in general not satisfied by the *spectral* radius. It is also worthwhile noting that if a Lanczos-type algorithm [3, 8] is used to derive the reduced-order matrix $\tilde{\boldsymbol{A}}$, a guaranteed-stability result similar to Theorem 3.1 is in general not possible. This is because the matrices $\boldsymbol{V}_q$ and $\boldsymbol{W}_q$ produced by the Lanczos algorithm such that $\boldsymbol{V}_q^T\boldsymbol{A}\boldsymbol{W}_q = \boldsymbol{H}_q$ do not allow us to conclude that $r(\boldsymbol{H}_q) \leq r(\boldsymbol{A})$.

Finally, we remark that the sufficient condition of Theorem 3.1 is not very conservative. Indeed, if $\boldsymbol{A} = \alpha\boldsymbol{I}_n$, with $|\alpha| < 1$, and $\boldsymbol{b} = \boldsymbol{e}_1$, then for any $q \leq n$, we have

$$\rho(\tilde{\boldsymbol{A}}) = r(\tilde{\boldsymbol{A}}) = r(\boldsymbol{A}) = \frac{1}{2}(||A||_1 + ||A||_\infty) = |\alpha| < 1.$$

Note that this is also the case for the unsymmetric matrices $\alpha\boldsymbol{P}_{n,q}$, where the last $n-q$ columns of $\boldsymbol{P}_{n,q}$ are permutations of the last $n-q$ columns of the identity $\boldsymbol{I}_n$.

4. CONCLUSION

The contributions of this paper are twofold. First, we have used the Arnoldi algorithm to show how a reduced-order model can be obtained without explicitly solving the ill-conditioned Hankel linear system of the classical Padé approximation procedure. Then we have established some results regarding the guaranteed stability of the reduced model. Although this paper has addressed only single-input/single-output systems, all our conclusions remain valid for the multiple-input/multiple-output case. Indeed, the block Arnoldi algorithm [4] could be used to obtain the reduced-order model, while the stability results continue to hold since they depend solely on Equation (4) and the system matrix $\boldsymbol{A}$.

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