Efficient Model Reduction of Interconnect via Approximate System Gramians

Jing-Rebecca Li Jacob White
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139
Email: {jingli, white}@rle-vlsi.mit.edu

Abstract

Krylov-subspace based methods for generating low-order models of complicated interconnect are extremely effective, but there is no optimality theory for the resulting models. Alternatively, methods based on truncating a balanced realization (TBR), in which the observability and controllability gramians have been diagonalized, do have an optimality property but are too computationally expensive to use on complicated problems. In this paper we present a method for computing reduced-order models of interconnect by projection via the orthogonalized union of the approximate dominant eigenspaces of the system's controllability and observability gramians. The approximate dominant eigenspaces are obtained efficiently using an iterative Lyapunov equation solver, Vector ADI, which requires only linear matrix-vector solves. A spiral inductor and a transmission line example are used to demonstrate that the new method accurately approximates the TBR results and gives much more accurate wideband models than Krylov subspace-based moment matching methods.

Keywords: Model Reduction, Truncated Balanced Realization, Lyapunov Equation, Vector ADI, Krylov Subspace

1 Introduction

The need to accurately model interconnect and packaging in circuit-level simulators has led to the development of a variety of robust approaches for generating low-order models of interconnect. The most popular approach for computing these low-order models, either directly from 3-D simulation or from extracted RLC circuits, is based on moment-matching via numerically robust orthogonalized Krylov subspace methods [1, 12, 3, 10, 9]. An alternative, the Truncated Balanced Realization methods (TBR) [4, 11], have never been given serious consideration even though they generate near-optimal reduced order models with a known L^∞-transfer function error bound. The difficulty with TBR methods is that they require the solution of two Lyapunov equations and then a full singular value decomposition, and are too computationally expensive to use on complicated interconnect problems.

In this paper we describe an approach to model reduction which attempts to approximate Truncated Balanced Realization cheaply. The technique presented uses the recently developed Vector ADI [7] algorithm for computing approximations to the dominant eigenspace of matrices that satisfy the Lyapunov equation. Then, the reduced order model is constructed by forming the orthogonalized union of the two dominant eigenspaces derived from solving the Lyapunov equations for the controllability and observability gramians. This is different from [7], in which only the dominant controllable subspace is used. In addition, we use Vector ADI to obtain an approximation of a higher rank than the desired reduction order, and then use a subspace in the projection. This results in a more accurate reduced model than in [7].

Section 2 gives brief background on model reduction. Section 3 describes using Vector ADI to obtain an approximate dominant gramian eigenspace. In section 4 we describe the new model reduction algorithm. We prove in section 5 the equivalence to TBR in a special case. In section 6 two numerical examples are used to compare the new approach with TBR and moment matching via Lanczos. Section 7 contains concluding remarks and acknowledgements.

2 Model Reduction

A linear time-invariant system with realization (A, B, C) is characterized by the equations:

$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx \quad (2)$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{p\times 1}$, and $y \in \mathbb{R}^{q\times 1}$ are the vector of state variables, inputs, and outputs, respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, are the system matrix, the input coefficient matrix, and the output coefficient matrix, respectively. It is assumed that p and q are both very small compared to the number of state variables n.

The system has controllability gramian P and observability gramian Q, which are symmetric, positive definite, and satisfy the following Lyapunov equations

$$AP + PA^T + BB^T = 0 \quad (3)$$
$$A^T Q + QA + C^T C = 0 \quad (4)$$

The gramians are needed in optimal Hankel-norm or Truncated Balanced Realization-type model reductions[4, 11].

The system described by equations (1-2) is characterized by its transfer function $G(s)$,

$$G(s) = C(sI - A)^{-1}B, \quad Y(s) = G(s)U(s). \quad (5)$$

Model order reduction seeks to obtain a smaller system such that the number of state variables of this new systems is much smaller than n, and the transfer function of the new system is close to the original.

2.1 Moment Matching Methods

Krylov subspace-based moment matching methods [5, 6] usually utilize the Arnoldi or Lanczos method to find an orthonormal basis for some combination of Krylov subspaces, $\mathcal{K}_J(A,B)$, $\mathcal{K}_J(A^T,C^T)$, $\mathcal{K}_J((A-pI)^{-1},B)$, or $\mathcal{K}_J((A^T-pI)^{-1},C^T)$, where

$$\mathcal{K}_J(A,B) = span\{B, AB, A^2B, \cdots, A^{(J-1)}B\}. \quad (6)$$

Projection of A onto an union of these Krylov subspaces results in a reduced system whose transfer function moments match those of the original system up to a certain order [5].

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Moment-matching methods require only matrix-vector products or solves, and hence are very efficient. However, there is no theoretical error bound for the reduced system's transfer function. The error will be small at points where moments were matched, but there is no guarantee that the error will also be small elsewhere.

2.2 Truncated Balanced Realization

Truncated Balanced Realization produces a guaranteed stable reduced system and has a theoretical transfer function error bound. The following summarizes the development in [4].

Given a stable system described by equations (1-2), with controllability and observability gramians, P and Q, respectively. Let Q have a factorization $Q = R^T R$, then RPR^T will be positive-definite and can be diagonalized as

$$RPR^T = U\Sigma^2 U^T, \tag{7}$$

with $U^T U = I$ and $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n > 0$ are the singular values of RPR^T. A balancing transformation is given by $T = \Sigma^{-1/2} U^T R$.

In the transformed state space coordinates, with realization $(A_b = TAT^{-1}, B_b = TB, C_b = CT^{-1})$, the new controllability and observability gramians are diagonal and equal, $P_b = Q_b = \Sigma = diag\{\sigma_1, \sigma_2, \cdots, \sigma_k, \sigma_{k+1}, \cdots, \sigma_n\}$.

If $\sigma_k > \sigma_{k+1}$, then the kth order truncated balanced realization is given by

$$(A^k_{tbr}, B^k_{tbr}, C^k_{tbr}) = (A_{11}, B_1, C_1) \tag{8}$$

where $A_{11} \in \mathbb{R}^{k \times k}, B_1 \in \mathbb{R}^{k \times p}, C_1 \in \mathbb{R}^{q \times k}$ are principal submatrices of the balanced realization, (A_b, B_b, C_b). The resulting transfer function $G^k_{tbr}(s)$ has L^∞-error

$$\|G(jw) - G^k_{tbr}(jw)\|_{L^\infty} \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n). \tag{9}$$

3 Vector ADI

Vector ADI was developed in [7] to provide a low-rank approximation to the solution of the Lyapunov equation with a low rank right hand side, and is derived from the full Alternate Direction Implicit method [2, 8]. Dominant eigenspace information tends to emerge quickly in Vector ADI, even if the full solution error is not yet small. The following summarizes the development in [7].

VADI iterates on the matrix square root V_J of the approximate solution $\tilde{X}$, ($\tilde{X} = V_J V_J^T$), to $AX + XA^T + BB^T = 0$. The number of iterations needed to achieve a required error tolerance is determined a priori [8]. Then the ADI parameters $\{p_j\}$ are calculated as a function of the required number of iterations and A's spectral bounds.

If the number of iterations to be performed is J, then $V_J = vadi(A, B, J)$ is:

$$\begin{aligned} V_J &= [w_J, P_{J-1} w_J, \cdots, P_1 P_2 \cdots P_{J-1} w_J] && (10) \\ w_J &= \sqrt{2p_J}(A - p_J I)^{-1} B && (11) \\ P_l &= \frac{\sqrt{2p_l}}{\sqrt{2p_{l+1}}}[I + (p_{l+1} + p_l)(A - p_l I)^{-1}] && (12) \end{aligned}$$

The starting vector w_J is obtained from a linear matrix-vector solve, and each succeeding p-vector of V_J is obtained from the previous one at the cost of a linear matrix-vector solve. The columns of V_J span a rational Krylov subspace, $\mathcal{K}(w_J, \mathbf{P}(A), J)$.

The Vector ADI approximation is then $\tilde{X}_J = V_J V_J^T$, which has rank Jp and error bounded by

$$\begin{aligned} &\|\tilde{X}_J - X\|_F \leq \|T\|_2^2 \|T^{-1}\|_2^2 k(\mathbf{p})^2 \|X\|_F, \\ &k(\mathbf{p}) = \max_{x \in spec(A)} |\prod_{j=1}^{J} \frac{(p_j - x)}{(p_j + x)}|, \end{aligned} \tag{13}$$

where T is a matrix of eigenvectors of A, and $\mathbf{p} = \{p_1, p_2, \cdots, p_J\}$ are the ADI parameters.

If the Lyapunov solution X has the singular value decomposition $X = U\Sigma U^T$, $U = [U^k, U^{n-k}]$, with diagonal of $\Sigma = diag(\sigma_1, \cdots, \sigma_n)$, in decreasing order, then U_k is the k-dim dominant eigenspace of X with associated eigenvalues (also singular values) $\sigma_1, \cdots, \sigma_k$. If $\tilde{X}$ is close to the exact solution X, a good approximation to U_k is given by the k-dim dominant eigenspace $\tilde{U}_k$ of $\tilde{X}$. In practice, $\tilde{U}_k$ obtained by Vector ADI tends to line up quickly with U_k.

The singular value decomposition of $V_J = vadi(A, B, J) = U_J \Lambda_J W_J^T$ can be obtained cheaply because V_J contains only Jp vectors. If $k \leq J$ and $U_J = [U_k, U_{J-k}]$, then U_k is the dominant eigenspace of $\tilde{X} = V_J V_J^T$, with associated eigenvalues $diag(\Lambda_J(1:k, 1:k)) = \lambda_1^2, \cdots, \lambda_k^2$.

4 Reduction via Union of Dominant Gramian Eigenspaces

Because balancing the gramians require complete knowledge of the entire eigenspace of both gramians, it is not in general possible to approximate TBR without good approximation to the full eigenspaces of both gramians.

Since only the dominant eigenspaces of the controllability and observability gramians are obtainable cheaply through Vector ADI, we propose a model reduction method which utilizes all the available information. We propose projecting the original system onto the orthogonalized union of the two dominant eigenspaces.

Algorithm:

1. Choose J and let $V_J^{ct} = vadi(A, B, J)$ and $W_J^{ob} = vadi(A^T, C^T, J)$.
2. Calculate SVD of V_J^{ct} and W_J^{ob}. $V_J^{ct} = U_J^{ct} \Lambda_J^{ct} (U_J^{ct})^T$ and $W_J^{ob} = U_J^{ob} \Lambda_J^{ob} (U_J^{ob})^T$
3. Choose $k \leq J$ and let $U_r^m = gram-schmidt[U_J^{ct}(:, 1:k), V_J^{ob}(:, 1:k)]$. Note $k \leq rank(U_r^m) = m \leq 2k$.
4. Reduce the system: $A_r^m = U_r^{mT} A U_r^m, B_r^m = (U_r^m)^T B, C_r^m = C U_r^m$.

Remarks *J may be much larger than k if A is poorly conditioned. To prevent ill-conditioning in forming the rational Krylov space in (10), back orthogonalization can be performed inside VADI. Then V_J^{ct} is stored as its QR decomposition, $V_J^{ct} = Q_J R_{J \times J}$.*

5 A Special Case

If the k most controllable modes span the same space as the k most observable modes, the kth-order TBR reduction can be obtained by projection via the k-dim dominant eigenspace of either gramian, without having to calculate the entire coordinate transformation T.

Theorem 1 *Let the gramians P and Q have SVD, $P = U_p \Sigma_p U_p^T, U_p = [U_p^k, U_p^{n-k}]$, and $Q = U_q \Sigma_q U_q^T, U_q = [U_q^k, U_q^{n-k}]$. Let $(A^k_{tbr}, B^k_{tbr}, C^k_{tbr})$ be the kth-order TBR reduction, with the factorization $Q = R^T R$ given by $R = \Sigma_q^{1/2} U_q^T$. Let $A_r^k =$*

$(U_q^k)^T A U_q^k$, $B_r^k = (U_q^k)^T B$, $C_r^k = C U_q^k$ *be the reduction by Q's dominant eigenspace. If* $span(U_p^k) = span(U_q^k)$, *then*

$$C_r^k(sI - A_r^k)^{-1} B_r^k = C_{tbr}^k (sI - A_{tbr}^k)^{-1} B_{tbr}^k \qquad (14)$$

Proof:

1. $U_q^T U_p = \begin{pmatrix} U_{pq}^k & 0 \\ 0 & U_{pq}^{n-k} \end{pmatrix}$ is $(k, n-k)$-block diagonal and both blocks are themselves unitary.

2. In equation (7), $RPR^T = \begin{pmatrix} W_{pq}^k & 0 \\ 0 & W_{pq}^{n-k} \end{pmatrix}$, where $W_{pq}^k = (\Sigma_q^k)^{1/2} U_{pq}^k \Sigma_p^k (U_{pq}^k)^T (\Sigma_q^k)^{1/2}$, and $W_{pq}^{n-k} = (\Sigma_q^{n-k})^{1/2} U_{pq}^{n-k} \Sigma_p^{n-k} (U_{pq}^{n-k})^T (\Sigma_q^{n-k})^{1/2}$.

3. Let $W_{pq}^k = U^k (\Sigma^k)^2 (U^k)^T$ and $W_{pq}^{n-k} = U^{n-k} (\Sigma^{n-k})^2 (U^{n-k})^T$ be SVDs, then $U = \begin{pmatrix} U^k & 0 \\ 0 & U^{n-k} \end{pmatrix}$, which is unitary, and $\Sigma = \begin{pmatrix} \Sigma^k & 0 \\ 0 & \Sigma^{n-k} \end{pmatrix}$ can be the SVD of RPR^T in (7).

4. $T = \Sigma^{-1/2} U^T R = \begin{pmatrix} S^k & 0 \\ 0 & S^{n-k} \end{pmatrix} \begin{pmatrix} (U_q^k)^T \\ (U_q^{n-k})^T \end{pmatrix}$, where S is invertible. Then $A_{tbr}^k = S_k (U_q^k)^T A U_q^k S_k^{-1}$, $B_{tbr}^k = S_k (U_q^k)^T B$, $C_{tbr}^k = C U_q^k S_k^{-1}$, and

$$\begin{aligned} & C_{tbr}^k (sI - A_{tbr}^k)^{-1} B_{tbr}^k \\ = \; & C U_q^k S_k^{-1} (sI - S_k (U_q^k)^T A U_q^k S_k^{-1})^{-1} S_k (U_q^k)^T B \\ = \; & C U_q^k S_k^{-1} S_k (sI - (U_q^k)^T A U_q^k)^{-1} S_k^{-1} S_k (U_q^k)^T B \\ = \; & C U_q^k (sI - (U_q^k)^T A U_q^k)^{-1} (U_q^k)^T B \\ = \; & C_r^k (sI - A_r^k)^{-1} B_r^k \end{aligned}$$

□

6 Numerical Results

The new model reduction method was compared with TBR and moment matching around $s = 0$ via Lanczos. The first example comes from inductance extraction of an on-chip planar square spiral inductor suspended over a copper plane [6]. The original system is order 500 and symmetric, so only one Lyapunov equation is solved. The relative inductance errors of the different models are shown in Figures 1-2.

Figure 1 compares reductions of order 7. VADI-7-11 comes from projection via the 7-dim dominant eigenspace obtained by 11 VADI iterations. VADI-7-12 uses 12 VADI iterations. MMVA-7 is order 7 moment matching around $s = 0$ via Arnoldi.

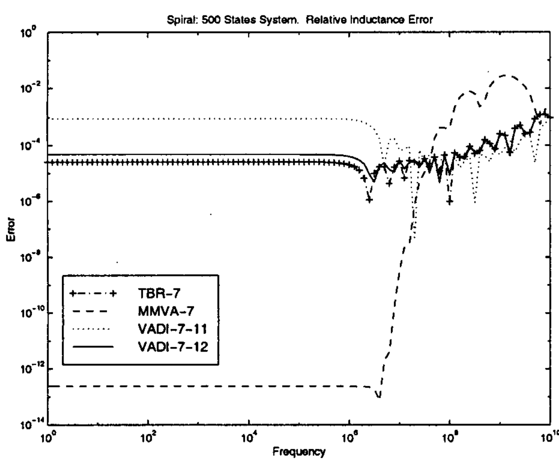

Fig 1: Spiral inductor inductance error.

It can be seen in Figure 1 that running one more iteration of Vector ADI reduces the error by more than one order of magnitude. VADI-7-12 is a very good approximation to TBR-7 and both have flat error over the entire frequency range, unlike MMVA-7 which has almost no error near $s = 0$ and large error far away.

Figure 2 compares order 13 moment matching (MMVA-13) with VADI-7-12. Note that though both require the same number of matrix-vector solves, VADI-7-12 is a smaller reduced system, order 7 versus MMVA-13's order 13.

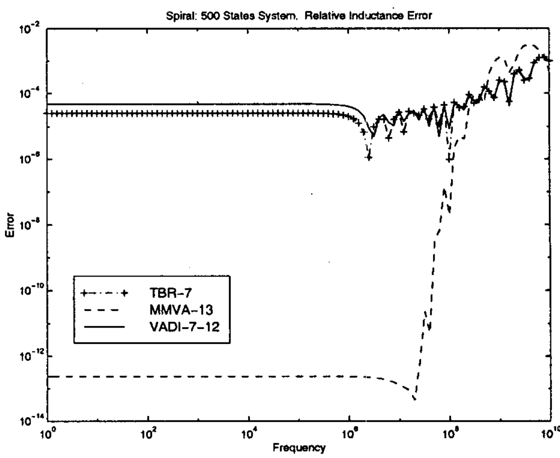

Fig 2: Inductance error.

VADI-7-12's L^∞-error is about half an order of magnitude smaller than MMVA-13's.

The spiral inductor has relatively simple and smooth frequency response behavior, which makes it easy to model by both VADI and MMVA.

An example that exhibits more complicated behavior comes from the discretization of a transmission line using the formulation in [9], with the original system having 256 states. The system matrix is not symmetric and it illustrates the general case when the dominant eigenspaces of the two gramians are different.

Figures 3 compares projection by the union of the exact dominant eigenspaces (CTOB) with Truncated Balanced Realization. Both reductions are order 10. CTOB-10 uses the union of the two exact 5-dim dominant eigenspaces.

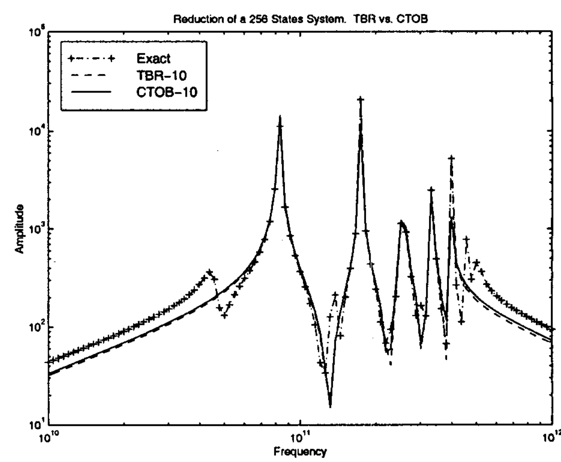

Fig 3: Transmission Line: CTOB close to TBR

For this transmission line example, projection by the union of the dominant eigenspaces produces a reduced model that is almost indistinguishable from TBR.

Figure 4 compares CTOB-10 with projection by either the 10-dim dominant controllable subspace only (CT-10) or the 10-dim dominant observable subspace only (OB-10).

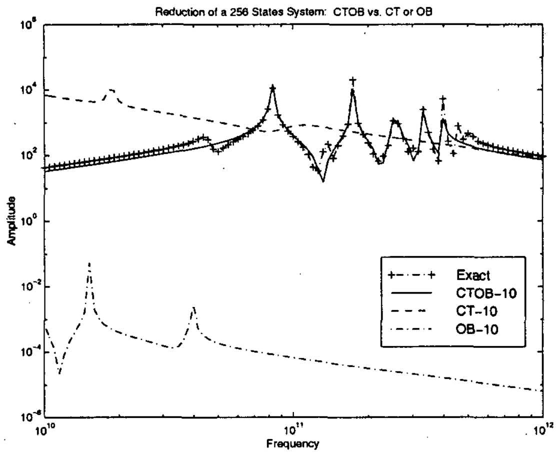

Fig 4: CTOB better than either CT or OB

Neither CT-10 nor OB-10 alone comes close to capturing the frequency response behavior.

Figure 5 compares the new method, using the approximate dominant eigenspaces calculated via Vector ADI (ADIctob), with moment matching via Lanczos (MMlanz). MMlanz-18 requires 34 matrix-vector solves, ADIctob-10(15), where the two 5-dim dominant gramian eigenspaces are each obtained after 15 VADI iterations, requires 30 matrix-vector solves.

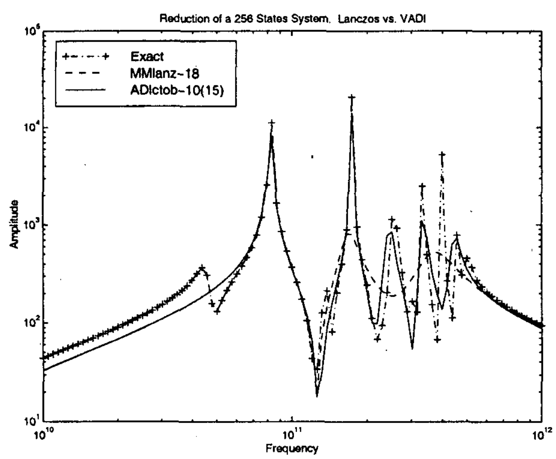

Fig 5: ADIctob captures global behavior.

ADIctob-10(15) clearly captures the global frequency response behavior much better than MMlanz-18. It captured all but the next to last sharp peak and averages the first tiny peak and a couple of small bumps between sharp peaks. This keeps the L^∞-error small without having to follow every topographical feature exactly. MMlanz-18 completely loses accuracy after the first sharp peak.

7 Conclusions and Acknowledgements

In this paper we presented a new method of model reduction of interconnect via projection onto the orthogonalized union of the approximate dominant controllable and observable subspaces, which are obtained through an iterative Lyapunov equation solver, Vector ADI. This new method is as inexpensive as Krylov space-based moment matching methods. It approximates Truncated Balanced Realization in the special case when the most controllable modes and the most observable modes span the same subspace. Two numerical examples show that the new method captures global frequency response behavior much better than the moment matching methods, and offers the flexibility of keeping the reduced model order low even when making higher order approximations.

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