Multiplierless Multiple Constant Multiplication

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Abstract. A variable can be multiplied by a given set of fixed-point constants using a multiplier block that consists exclusively of additions, subtractions, and shifts. The generation of a multiplier block from the set of constants is known as the multiple constant multiplication (MCM) problem. Finding the optimal solution, namely, the one with the fewest number of additions and subtractions, is known to be NP-complete. We propose a new algorithm for the MCM problem, which produces solutions that require up to 20% less additions and subtractions than the best previously known algorithm. At the same time our algorithm, in contrast to the closest competing algorithm, is not limited by the constant bitwidths. We present our algorithm using a unifying formal framework for the best, graph-based MCM algorithms and provide a detailed runtime analysis and experimental evaluation. We show that our algorithm can handle problem sizes as large as 100 32-bit constants in a time acceptable for most applications. The implementation of the new algorithm is available at www.spiral.net.

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1. *Introduction*

This article addresses the problem of efficiently computing a set of products $t_i x$, for $i = 1, \dots, n$, of a variable x with several known fixed-point constants t_i *multiplierless*, that is, using exclusively additions, subtractions, and shifts. This problem is

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known as multiple constant multiplication (MCM). Avoiding costly multipliers is particularly important in hardware implementations, for example, of digital signal processing functionality such as filters or transforms. However, replacing constant multiplications with additions and shifts can also be relevant in software implementations, for example, as optimization for speed, since integer multipliers often have a significantly lower throughput than adders, but also for embedded processors, which may not feature a multiplication unit at all. The MCM problem can be considered as a fundamental problem in computer arithmetic.

We propose a new algorithm for the MCM problem, which generates solutions that are significantly better—in terms of the number of additions/subtractions of the solution—than any of the previously published algorithms, that is at the same time more generally applicable. To more clearly state our contribution and put it in the context of previous work, we first introduce the problem in greater detail.

1.1. SINGLE CONSTANT MULTIPLICATION (SCM). The multiplication $y = tx$ of a variable x by a known integer or fixed-point constant t can be decomposed into additions (adds), subtractions (subtracts), and binary shifts. The problem of finding the decomposition with the least number of operations is known as the *single constant multiplication (SCM)* problem, and is NP-complete, as shown in Cappello and Steiglitz [1984]. Without loss of generality we will assume that the constants are integers, since a fixed-point multiplication is equivalent to a multiplication by an integer followed by a right shift. The SCM problem is related to but different from the addition chain problem [Knuth 1969], which multiplies by a constant using additions only. The permission of shifts fundamentally alters the problem as well as the strategies for its solution.

The straightforward method for decomposing the multiplication into adds and shifts translates 1's in the binary representation of the constant t into shifts, and adds up the shifted inputs. For example, for $t = 71$,

$$71x = 1000111_2 x = x \ll 6 + x \ll 2 + x \ll 1 + x,$$

which requires 3 adds. Alternatively, the multiplication can be decomposed into subtracts and shifts by translating 0's into shifts, and subtracting from the closest constant consisting of 1's only (i.e., of the form $2^n - 1$).

$$71x = 1000111_2 x = (x \ll 7 - x) - x \ll 5 - x \ll 4 - x \ll 3$$

Taking the best of these two methods yields in the worst and average cases a solution with $\frac{b}{2} + O(1)$ adds/subtracts, where b is the bitwidth of t.

A better digit-based method decomposes into both adds and subtracts by recoding the number into the canonical signed digit (CSD) representation [Avizienis 1961], which allows negative digits $\overline{1}$. Using CSD, the previous example can be improved to use only 2 add/subtract operations.

$$1000111_2 x = 100100\overline{1}_{\text{CSD}} x = x \ll 6 + x \ll 3 - x$$

Using CSD, the worst-case cost remains $\frac{b}{2} + O(1)$, but the average case is now improved to $\frac{b}{3} + O(1)$ [Wu and Hasan 1999].

The optimal decomposition in terms of add/subtract operations is in general not obtained with CSD, and its worst-case and average costs are unknown. Dempster and Macleod [1994] designed an exhaustive search algorithm to find the optimal decompositions for constants up to 12 bits. The authors also showed that using shifts

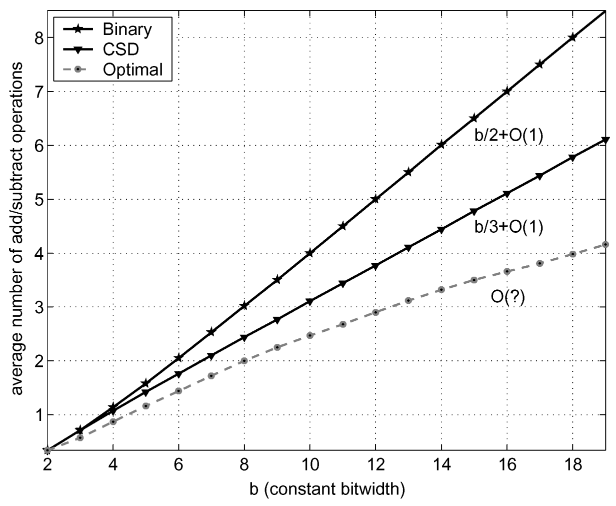
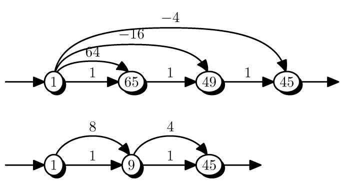


FIG. 1. Average number of add/subtract operations required for multiplying by a constant of different bitwidths.



$$65x = x + 64x$$
$$49x = 65x - 16x$$
$$45x = 49x - 4x$$

$$9x = 8x + x$$
$$45x = 5(9x) = 9x + 4(9x)$$

FIG. 2. Multiplication by 45 using 3 adds/subtracts (CSD, top) and 2 adds/subtracts (optimal, bottom). The vertices represent add/subtract operations labeled with their outputs, and the edges represent shifts labeled with the corresponding scaling (a 2-power). Negative scaling indicates that a subtraction is performed.

no larger than $b + 1$ is sufficient to yield optimal solutions for 12-bit constants. Their work has been extended by Gustafsson et al. [2002] to constants up to 19 bits, again yielding optimal results regardless of shift constraints. Although the asymptotic worst-case cost of the optimal decomposition remains an open research problem, it appears to be asymptotically better than $O(b)$, as shown in Figure 1. The plot compares the three decomposition methods by showing the average number of adds/subtracts (y axis) obtained for 300 uniformly distributed random constants of bitwidths from 2 to 19 (x axis).

Consider the smallest constant for which the CSD decomposition is suboptimal, namely, 45. Figure 2 shows its CSD (three add/subtract operations) and optimal (two add/subtract operations) decompositions both visualized as graphs. We observe that the optimal decomposition uses a different graph topology than the CSD decomposition. Intuitively, digit-based methods such as CSD produce suboptimal results because they only consider one type of graph topology. The exhaustive search methods in Dempster and Macleod [1994] and Gustafsson et al. [2002], on the other hand, consider all possible graph topologies to find optimal decompositions.

1.2. MULTIPLE CONSTANT MULTIPLICATION (MCM). An extension of SCM is the problem of multiplying a variable x by several constants $t_1, \ldots, t_n$ in parallel

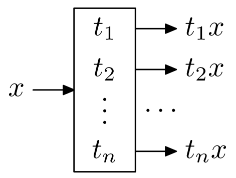


FIG. 3. (Multiple constant) multiplier block.

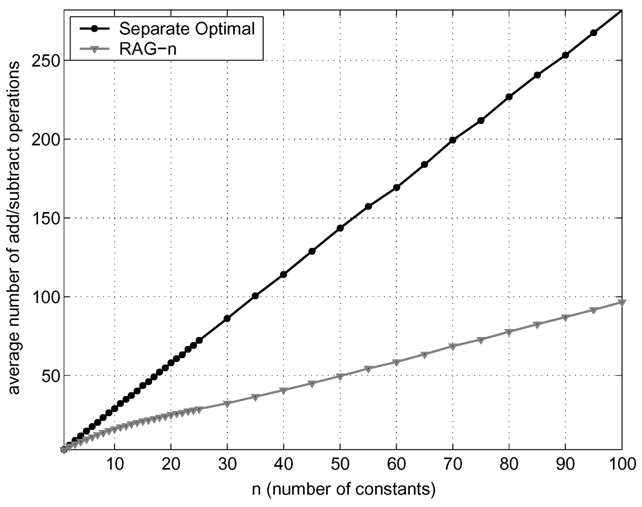


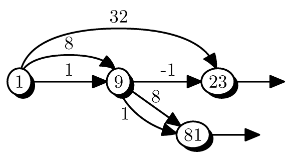
FIG. 4. Average number of add/subtract operations required for multiplying by each of the 12-bit coefficients in a set of given size.

in a so-called *multiplier block*, shown in Figure 3. Since intermediate results of the constant decompositions may be shared, a multiple constant multiplier block may be decomposed into fewer operations than the sum of the single constant decompositions' operation counts. The problem of finding the decomposition with fewest operations is known as *multiple constant multiplication (MCM)*.

The potential savings from sharing intermediate results increase with the number of constants, which is illustrated in Figure 4. The plot compares the number of add/subtract operations (y axis) for varying sizes n (x axis) of sets of 12-bit constants using separate optimal SCM decompositions and using RAG-n, the heuristic MCM algorithm from Dempster and Macleod [1995]. Since MCM is a generalization of SCM, it is also NP-complete.

The MCM problem is particularly relevant for the multiplierless implementation of digital finite impulse response (FIR) filters [Bull and Horrocks 1991], but also for matrix-vector products with a fixed matrix, including linear signal transforms [Püschel et al. 2004; Chen et al. 2002; Liang and Tran 2001] such as the discrete Fourier transform or discrete cosine transform, for example. In an n-tap FIR filter, every input sample is multiplied by all n taps. Discrete Fourier and trigonometric transform algorithms, on the other hand, involve 2×2 rotations, which require simultaneous multiplication by two constants.

Figure 5 is an example of a multiplier block which implements the parallel multiplication by 23 and 81 using only 3 add/subtract operations, although the separate optimal decompositions of 23 and 81 each require 2 add/subtract operations.



$$
\begin{aligned}
9x &= 8x + x\\
23x &= 32x - 9x\\
81x &= 8(9x) + 9x
\end{aligned}
$$

FIG. 5. Multiplier block with constants 23 and 81.

The different problem of *multiplexed* multiple constant multiplication was considered in Tummeltshammer et al. [2004]. In this case, the multiplier block contains multiplexers that are switched by control logic to achieve multiplication by different constants; this way sequential multipliers can be fused. We will not consider this problem in this article.

Existing Algorithms. The existing MCM algorithms can be divided into four general classes:

—digit-based recoding;

—common subexpression elimination (CSE) algorithms;

—graph-based algorithms; and

—hybrid algorithms.

Digit-based recoding includes simple methods like CSD and the binary method mentioned earlier. They generate the decomposition directly from the digit representation of the constant. These methods are the fastest and the worst-performing; however, newer work by Coleman [2001] uses different number systems to yield considerably better solutions. The main advantage of digit-based recoding is their low computational cost, typically linear in the number of bits. As a consequence, these methods can be easily applied to constants with thousands of bits.

Common subexpression elimination (CSE) algorithms are direct descendants of digit-based recoding methods. The basic idea is to find common subpatterns in representations of constants after the constants are converted to a convenient number system such as CSD. Examples for this method include Pasko et al. [1999], Lefèvre [2001], and Hartley [1996]. The disadvantage, however, is that the performance of these algorithms depends on the number representation. Further, even though the considered CSE problem is NP-complete [Garey and Johnson 1979; Downey et al. 1980], its optimal solution does in general not provide the optimal MCM solution. More recently, Dempster and Macleod [2004] have proposed searching over alternative number representations to find considerably improved solutions using a CSE algorithm.

Graph-based algorithms are bottom-up methods that iteratively construct the graph (as in Figure 5) representing the multiplier block. The graph construction is guided by a heuristic that determines the next graph vertex to add to the graph. Graph-based algorithms offer more degrees of freedom by not being restricted to a particular representation of the coefficients, or a predefined graph topology (as in digit-based algorithms), and typically produce solutions with the lowest number of operations. Examples of graph-based algorithms include Bull and Horrocks [1991], RAG-n [Dempster and Macleod 1995], and Bernstein [1986]. This article proposes a new graph-based algorithm.

Hybrid algorithms combine different algorithms, possibly from different classes. For example, Choo et al. [2004] construct the multiplier block graph with fixed topology to compute the so-called *differential coefficients*, and then switch to a CSE algorithm for multiplication by the differential coefficients.

Today, to the best of our knowledge, RAG-n yields solutions with the smallest number of add/subtract operations among all algorithms. Graph-based algorithms are expected to outperform other methods, since they have the fewest restrictions. However, RAG-n relies on a lookup table of optimal single constant decompositions, which is currently limited to 19 bits, as mentioned before.

Contribution of This Article. This article first presents a general formal framework that captures the common structure of graph-based MCM algorithms. A crucial component in this framework is our notion of "$\mathcal{A}$-distance," an extension of the concept of adder distance introduced in Dempster and Macleod [1995], and its exact or heuristic estimation. We use the framework to develop a new graph-based MCM algorithm that outperforms the best available algorithms with respect to the number of add/subtract operations in the obtained multiplier blocks. In particular, we achieve up to a 20% lower average operation count than the best previous algorithm, RAG-n. At the same time, our new algorithm is not bitwidth limited like RAG-n, and can thus be used to generate multiplier blocks for all practically relevant bitwidths. Finally, we perform a detailed runtime analysis of our new algorithm and other graph-based algorithms used for benchmarks. This analysis was not provided in the original papers.

Other Optimization Metrics. Besides reducing the number of add/subtract operations, it is often desirable to optimize for other metrics, for example, the critical path of the MCM block, or the register pressure in the generated code. Examples of such work include Dempster et al. [2002] and Kang et al. [2001]. This article does not consider this type of optimization; however, the structure of our algorithm enables its adaptation to other target metrics.

Organization. This article is organized as follows. Section 2 gives the mathematical background and notation used throughout the article. Section 3 puts existing graph-based algorithms for the MCM problem into a framework that identifies their common structure. Our new algorithm is presented in Section 4, including a discussion of its main properties. A crucial subroutine of our algorithm estimates the so-called $\mathcal{A}$-distance, which is explained in Section 5. Section 6 presents a runtime analysis of both our and competing algorithms and shows various experimental results with generated MCM blocks. The results show that our algorithm outperforms the best available methods at the cost of a higher computation time. Finally, we offer conclusions in Section 7.

2. *Background*

In this section we formally state the problem of multiple constant multiplication, describe the graphical representation of multiplier blocks, and the corresponding mathematical notation used in this article. The notation introduced in this article is summarized in Table I and will be used in following sections to develop a unifying framework for existing and our new graph-based algorithms.

TABLE I. Summary of the Notation Used in the Article

Notation	Meaning	Defined in
A	Upper case letters denote sets	
a	Lower case letters denote integers	
$U \cup V$	Union of sets U and V	
$U - V$	Difference of sets U and V	Eq. (3)
$U - a$	Difference of sets U and $\{a\}$	Eq. (3)
$U + a$	Union of sets U and $\{a\}$	Eq. (4)
$a + b$	Addition of two integers	
$a - b$	Subtraction of two integers	
UV	Set of all products	Eq. (8)
$\frac{U}{V}$	Set of all integer quotients	Eq. (9)
$\mathrm{dist}(U, a)$	$\mathcal{A}$-distance of a from set U	Def. 2.4
$\mathcal{C}_n$	Set of complexity-n constants	Def. 2.3
$\mathcal{A}_p(a, b)$	$\mathcal{A}$-operation	Def. 2.1
$\mathcal{A}_*(a, b)$	Vertex fundamental set	Def. 2.5
$\mathcal{A}_*(U, V)$	Vertex fundamental set	Def. 2.6
$\mathcal{A}^{\mathrm{odd}}$	$\mathcal{A}$-operation with odd outputs	Sect. 2
R	Ready set	Sect. 3.1
S	Successor set of R	Eq. (10)
S^n	Set of distance-n constants with respect to R	Eq. (11)
T	Targets set	Sect. 3.1
$\mathrm{B}(R, s, t)$	Benefit	Eq. (15)
$\overline{\mathrm{B}}(R, s, t)$	Weighted benefit	Eq. (16)
$\mathrm{H}_{\mathrm{maxb}}(R, S, T)$	Maximum benefit heuristic	Sect. 4.3
$\mathrm{H}_{\mathrm{cub}}(R, S, T)$	Cumulative benefit heuristic	Sect. 4.3
$\mathrm{dist}(U, a) \simeq d$	d is an estimate for $\mathrm{dist}(U, a)$	Sect. 5.6
$\mathrm{Est}(z)$	Auxiliary cost measure of a constant z	Eq. (21)
$\mathrm{Est}(Z)$	Minimum auxiliary cost measure for a set Z	Eq. (22)

$\mathcal{A}$-Operation. A multiplier block implements the parallel multiplication by a given set of constants which we call *fundamentals*, following Dempster and Macleod [1994], or simply constants. The implementation uses adds, subtracts, and shifts, but to streamline the search process we consolidate these operations into a single parameterized operation called an *$\mathcal{A}$-operation*.

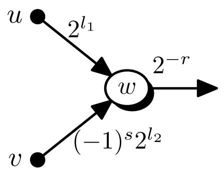
We define an $\mathcal{A}$-operation as an operation on fundamentals. An $\mathcal{A}$-operation performs a single addition or subtraction, and an arbitrary number of shifts which do not truncate nonzero bits of the fundamental. Since two consecutive shifts can be merged, the most general definition is given next.

Definition 2.1 (*General $\mathcal{A}$-Operation*). Let $l_1, l_2 \geq 0$ be integers (left shifts), $r \geq 0$ be an integer (right shift), and let $s \in \{0, 1\}$ (sign). An $\mathcal{A}$-operation is an operation with two integer inputs u, v (fundamentals) and one output fundamental, defined as

$$\begin{aligned} \mathcal{A}_p(u, v) &= |(u \ll l_1) + (-1)^s (v \ll l_2)| \gg r \qquad (1) \\ &= |2^{l_1} u + (-1)^s 2^{l_2} v| 2^{-r}, \end{aligned}$$

where $\ll$ is a left binary shift, and $\gg$ is a right binary shift, and $p = (l_1, l_2, r, s)$ is the *parameter set* or the *$\mathcal{A}$-configuration* of $\mathcal{A}_p$. To preserve all significant bits of the output, 2^r must divide $2^{l_1} u + (-1)^s 2^{l_2} v$.

Without loss of generality, we restrict the discussion to positive fundamentals only. The absolute value in the definition of $\mathcal{A}$, besides enforcing positive



$$p = (l_1, l_2, r, s)$$
$$w = \mathcal{A}_p(u, v) = |2^{l_1}u + (-1)^s 2^{l_2} v| 2^{-r}$$

FIG. 6. $\mathcal{A}$-operation: u and v are the input fundamentals and w is the output fundamental. $\mathcal{A}$-operations directly connected to the input of the multiplier block have $u = v = 1$.

fundamentals, enables the subtraction to be done in only one direction, which simplifies the definition.

We represent an $\mathcal{A}$-operation graphically as shown in Figure 6. Note that the result w is written inside the node, even though the right shift by r has already been applied to w. The reason for this notation is that what matters later in the considered MCM algorithms is the unique odd fundamental at each node. Also, the final right shift by r will be fused in the full $\mathcal{A}$-graph with a subsequent left shift in the next $\mathcal{A}$-operation. Originally, the final right shift r in the $\mathcal{A}$-operation was not used; however, it is necessary to obtain the full range of possible outputs. For example, for odd u and v and $l_1 = l_2 = 0$, $u + v$ is even and a right shift can be applied to obtain a new odd fundamental that may not be possible to obtain otherwise with only one $\mathcal{A}$-operation.

All MCM algorithms impose constraints on the $\mathcal{A}$-configuration. In particular, an upper bound on the left shifts l_1 and l_2 is necessary to make the set of choices finite. In practice, it is sufficient to limit these shifts by the bitwidth of the target constants. Experimental results from Dempster and Macleod [1994], Lefèvre [2003], and Gustafsson et al. [2002] show that allowing larger shifts does not improve upon the optimal solutions[1] obtained with the former limits. However, in the general case, the smallest obtained theoretical limit on the shifts sufficient to obtain optimal multiplier blocks is larger [Dempster and Macleod 1994]. Additional insights are given in Lefèvre [2003]. Although the authors do not compute this particular bound (c_{inf} in their notation), they prove that optimal multiplier blocks cannot have shifts larger than a certain finite value (c_{sup} in their notation). The results of Lefèvre [2003] and Dempster and Macleod [1994] are not directly comparable, since Lefèvre did not take into account right shifts.

Some constraints on p may lead to different problem classes. For example, if we restrict $l_1 = l_2 = r = s = 0$, then the SCM problem reduces to the well-known problem of finding the shortest addition chain [Knuth 1969], since the $\mathcal{A}$-operation in this case is an ordinary addition. However, even a lesser restriction, like $s = 0$, which disallows subtractions can require major changes to the MCM algorithm. Thus, it is crucial that the $\mathcal{A}$-operation and $\mathcal{A}$-configuration constraints are explicitly defined for any MCM algorithm.

Since in this work we are only interested in reducing the number of add/subtract operations in a multiplier block, we will refer to $\mathcal{A}$-operations also as add/subtract operations, although in reality they include also shifts.

The rationale for neglecting shifts is that in hardware, shifts can be implemented as wires which require virtually no area, and in software, some CPUs (e.g., Intel

[1] With respect to the number of add/subtract operations.

XScale) support combined add/shift and subtract/shift instructions which execute as fast as a single add or subtract. If desired, minimization of the number of shifts can be incorporated as a secondary criteria in our MCM algorithm.

$\mathcal{A}$-Graph. As we have already seen in Figures 2 and 5, the structure of a multiplier block can be represented as a directed graph. We call such graph an *$\mathcal{A}$-graph*, since it is built out of the $\mathcal{A}$-operations shown in Figure 6. The vertices of an $\mathcal{A}$-graph are labeled with their respective fundamentals; hence the input vertex has label 1. The edges are labeled with a 2-power scaling factor equivalent to the performed shift. Negative edge values are used to indicate subtractions at the following vertex.

Formal Problem Statement. Now we can formally state the problem of constructing multiplier blocks.

Definition 2.2 (*MCM Problem*). **Given** a set of positive target constants $T = \{t_1, \ldots, t_n\} \subset \mathbb{N}$. **Find** the smallest set $R = \{r_0, r_1, \ldots, r_m\}$ with $T \subset R$ such that $r_0 = 1$, and for all r_k with $1 \leq k \leq m$ there exist r_i, r_j with $0 \leq i, j < k$, and an $\mathcal{A}$-configuration p_k such that

$$r_k = \mathcal{A}_{p_k}(r_i, r_j). \tag{2}$$

The set of $\mathcal{A}$-graph fundamentals R and the set of $\mathcal{A}$-configurations p_k uniquely define an $\mathcal{A}$-graph for an MCM block with $m = |R| - 1$ add/subtract operations.

The number of $\mathcal{A}$-operations in an optimal solution for a given set T is called the *$\mathcal{A}$-complexity* of T. If any $\mathcal{A}$-graph for T is given, optimal or not, we refer to its number of required $\mathcal{A}$-operations as the cost of this graph.

All constants can be divided into complexity classes.

Definition 2.3 (*Complexity-n Constants*). We denote by $\mathcal{C}_n$ the set of all constants with complexity n, that is, those for which an optimal SCM solution requires exactly n $\mathcal{A}$-operations. For example, $\mathcal{C}_0 = \{2^a \mid a \geq 0\}$ because precisely all 2-power constants require a single left shift and no adds/subtracts.

Although the sets $\mathcal{C}_n$ are infinite, we will always limit our discussion to constants up to certain bitwidth b, which is always explicitly stated. The set of complexity n constants obeying this constraint is then finite and, by abuse of notation, will also be denoted $\mathcal{C}_n$.

Next, we introduce the notion of *$\mathcal{A}$-distance*, the key component in our proposed algorithm.

Definition 2.4 (*$\mathcal{A}$-Distance*). Let $c \in \mathbb{N}$ be a constant, and let $R \subset \mathbb{N}$ be a set of constants (fundamentals of an $\mathcal{A}$-graph). Then, the $\mathcal{A}$-distance of c from the set R, denoted $\text{dist}(R, c)$, is the minimum number of extra $\mathcal{A}$ operations required to obtain c, given R.

The $\mathcal{A}$-distance corresponds to the notion of "adder distance" in Dempster and Macleod [1995].

For example, $\text{dist}(\{1\}, T)$ is equal to the complexity of T, and for all $c \in \mathcal{C}_n$, $\text{dist}(\{1\}, c) = n$.

For simplicity, we write in this article the difference of two sets U and V as

$$U - V = U \backslash V, \quad \text{and for one element} \quad U - a = U \backslash \{a\}. \tag{3}$$

For visual appeal we will also use a $+$ for the union of a set with a single element, that is,

$$U + a = U \cup \{a\}. \tag{4}$$

To express the degree of freedom in the output of an $\mathcal{A}$-operation when different $\mathcal{A}$-configurations are chosen, we define next the *vertex fundamental set*.

Definition 2.5 (*Vertex Fundamental Set*). The set of all possible outputs (not equal to the inputs) of an $\mathcal{A}$-operation with fixed inputs (u and v) under different $\mathcal{A}$-configurations is called the *vertex fundamental set*, written as

$$\mathcal{A}_*(u, v) = \{\mathcal{A}_p(u, v) \mid p \text{ is a valid configuration } \} - u - v. \tag{5}$$

The definition of *valid* $\mathcal{A}$-configurations p will be introduced later for each discussed algorithm.

It is useful to extend the definition of $\mathcal{A}_*$ to sets of inputs.

Definition 2.6. If $U,\ V \subset \mathbb{N}$ are sets of fundamentals, then

$$\mathcal{A}_*(U, V) = \bigcup_{\substack{u \in U \\ v \in V}} \mathcal{A}_*(u, v) - U - V. \tag{6}$$

From this definition it follows that for sets U, V, and W:

$$\begin{aligned} \mathcal{A}_*(U \cup V, W) &= (\mathcal{A}_*(U, W) - V) \cup (\mathcal{A}_*(V, W) - U) \\ &= \mathcal{A}_*(U, W) \cup \mathcal{A}_*(V, W) - U - V \end{aligned} \tag{7}$$

Further, we define the product of sets U, V in the usual way as

$$UV = \{uv \mid u \in U,\ v \in V\}, \tag{8}$$

and their quotient as

$$\frac{U}{V} = \left\{\frac{u}{v} \mid u \in U,\ v \in V,\ v \text{ divides } u\right\}. \tag{9}$$

Odd Fundamental Graphs. Any $\mathcal{A}$-graph can be converted into an $\mathcal{A}$-graph of equal cost which has only odd fundamentals [Dempster and Macleod 1994]. Such graphs are called *odd fundamental graphs*. As an example, the graph in Figure 5 is an odd fundamental graph. This reduction is possible because any even constant can be obtained from an odd constant by a suitable shift. Odd fundamental graphs are important because they reduce the degree of freedom in choosing graph fundamentals, without affecting the cost of the graph.

To obtain odd fundamental graphs, the valid $\mathcal{A}$-configuration must be constrained to allow at most one nonzero left shift l_1 or l_2, and if $l_1 = l_2 = 0$, to force r to be the unique right shift that produces an odd value. We will sometimes use $\mathcal{A}^{\text{odd}}$ to indicate these constraints. For given u and v, the only free parameters in $\mathcal{A}^{\text{odd}}$ are s and the nonzero left shift (either l_1 or l_2), in contrast to the general $\mathcal{A}$-operation where l_1, l_2, r, and s can vary. Thus, the space of valid $\mathcal{A}$-configurations p is considerably reduced.

Algorithms that use $\mathcal{A}^{\text{odd}}$ preprocess all target constants with a suitable right shift to make them odd.

Algorithm 1 High-Level Structure of Graph-Based MCM Algorithms

Given the target set of constants T. Compute (synthesize) the set $R = \{r_1, \ldots, r_m\}$, with $T \subset R$, as given in Definition 2.2.

```
SynthesizeMultiplierBlock(T)
1: $R \leftarrow \{1\}$
2: while $T \neq \emptyset$ do
3:    compute the successor set $S$ of $R$
4:    select $s \in S$ based on a heuristic
5:    Synthesize($s$)

Synthesize(s)
1: $R \leftarrow R + s$
2: $T \leftarrow T - s$
```

3. *Overview of MCM Algorithms*

Using the notation introduced in the previous section, we put existing graphical MCM algorithms into a common context and identify their common structure. Then we discuss the most important algorithms in greater detail.

3.1. General Framework. Existing graph-based algorithms for multiplier block synthesis share the same high-level structure, that is shown in Algorithm 1 and explained in the following. As said before, it is necessary that the notion of $\mathcal{A}$-operation considered by the algorithm is precisely defined, including all constraints.

The input to Algorithm 1 is the *target set* T of constants. The set R used in the algorithm is called the *ready set*. It is initialized in step 1 with the first fundamental 1 and iteratively augmented in the loop in step 2 with additional fundamentals. Upon termination, namely, when $T \subset R$, then R is output as the solution. In each iteration of the loop in step 2, an element of the *successor set* S of R is chosen as the next fundamental based on a heuristic. Formally,

$$S = \{s \mid \text{dist}(R, s) = 1\} = \mathcal{A}_*(R, R) \qquad (10)$$

is the set of all constants of distance 1 from R. Even though S depends on R, we do not write S_R or $S(R)$ to simplify the notation. In Algorithm 1 we do not specify whether or how S is computed, which is discussed later.

Once $s \in S$ is chosen, it is added to R, and, if $s \in T$, removed from T. We call this process *synthesizing* s. Alternatively, the algorithm may use a heuristic to choose constants s at a higher distance.

$$s \in S^n = \{s \mid \text{dist}(R, s) = n\} \qquad (11)$$

In this case, all $n - 1$ intermediate fundamentals have to be synthesized as well. We call S^n the distance-n set (of R). Clearly, $S^1 = S$. Although S does not have to be explicitly enumerated and stored, some algorithms do so.

The procedure is repeated until the target set is empty, that is, all of the target constants are synthesized.

The heuristic used in step 4, which determines the next fundamental or the next vertex in the $\mathcal{A}$-graph to be synthesized, is highly dependent on the $\mathcal{A}$-operation used within an MCM algorithm. Further, the heuristic is what differentiates the various algorithms and determines their performance.

In the following we discuss the three most important graph-based MCM algorithms, each of which is an instantiation of Algorithm 1.

3.2. BULL-HORROCKS ALGORITHM (BHA). Bull and Horrocks [1991] designed four MCM algorithms: add, add/subtract, add/shift, and add/subtract/shift decompositions. Here we discuss the latter, since it addresses the problem considered in this work. We refer to this algorithm as BHA.

The $\mathcal{A}$-operation considered in BHA imposes constraints on the configuration by requiring $r = 0$, and $\mathcal{A}_p(u, v) \leq \min(T)$. In words, right shifts are not allowed and as intermediate fundamentals, only numbers smaller than the current $\min(T)$ are synthesized. This also imposes an implicit bound on the shifts l_1 and l_2 in the $\mathcal{A}$-configuration.

The heuristic synthesizes targets T in ascending order. Since targets are removed from T in the synthesize step, the next target to be synthesized is always $\min(T)$.

The heuristic used in BHA keeps track of the so-called "error"

$$\epsilon = \min(T) - \max(R). \tag{12}$$

If $\epsilon \in R$, then the candidate target can be directly synthesized, and the algorithm proceeds. Otherwise, two successors s_1 and s_2 that minimize the error are synthesized, chosen as follows:

$$\begin{aligned} s_1 &= \arg \min_{s \in S,\, s \leq \epsilon} (\epsilon - s), \text{ and} \\ s_2 &= \max(R) + s_1 \end{aligned}$$

In particular, when $\epsilon \in S$, then $s_1 = \epsilon$ and $s_2 = \epsilon + \max(R) = \min(T)$, that is, the candidate target is synthesized.

The algorithm considers only the magnitude of the error, and the binary representation of constants is not taken into account, unlike CSE algorithms.

3.3. BULL-HORROCKS MODIFIED ALGORITHM (BHM). Dempster and Macleod [1995] improved BHA and called it the *Bull-Horrocks modified algorithm (BHM).*

The $\mathcal{A}$-operation considered in BHM is $\mathcal{A}^{\text{odd}}$. Accordingly, all targets T are preprocessed by right shifts to become odd. The $\mathcal{A}$-configuration constraints are relaxed to allow fundamentals larger than constants in T, namely, up to $\mathcal{A}_p^{\text{odd}}(u, v) \leq 2\max(T)$, which stimulates the use of subtractions.

The heuristic in BHM is changed from BHA in the following way. First, the targets are synthesized in order of increasing $\mathcal{A}$-complexity, which is obtained from a precomputed lookup table, or estimated by, for example, the CSD cost. Second, the error in Eq. 12 is allowed to be negative. Finally, because $\mathcal{A}^{\text{odd}}$ is used and all elements of R are odd, minimization applies left shifts to candidate successors.

Let $\text{minc}(T)$ denote the next candidate target (target of minimal complexity or cost), and let $r_c \in R$ denote the closest (magnitude-wise) fundamental to $\text{minc}(T)$,

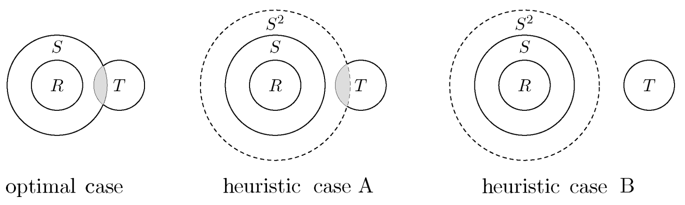


FIG. 7. The three cases considered by the RAG-n heuristic. The dashed circle means that S^2 is not computed explicitly.

namely, $r_c = \text{minc}(T) \pm \epsilon$. Then the chosen successors s_1 and s_2 are determined as follows.

$$\begin{aligned}(s_1, k) &= \arg \min_{\substack{s \in S \\ 0 \le k \le \lceil \log_2 \epsilon \rceil}} |\epsilon - s \ll k|, \\ s_2 &= r_c \pm s_1 \ll k\end{aligned} \tag{13}$$

Just as in BHA, two successors are synthesized per iteration unless $\epsilon \in R$.

Due to these modifications, BHM performs consistently better than BHA in terms of the number of $\mathcal{A}$-operations in the generated $\mathcal{A}$-graphs.

3.4. N-DIMENSIONAL REDUCED ADDER GRAPH (RAG-N). RAG-n [Dempster and Macleod 1995] is a graph-based MCM algorithm that requires a precomputed table of optimal SCM decompositions which are obtained by exhaustive search using the method in Dempster and Macleod [1994].

The target $\mathcal{A}$-operation in RAG-n is $\mathcal{A}^{\text{odd}}$, and as in BHM, all targets T are first right-shifted to become odd. The $\mathcal{A}$-configuration constraints are less restrictive than in BHM, with the only constraint being $\mathcal{A}_p(x, y) \le 2^{b+1}$, where b is the maximum bitwidth of the targets. The RAG-n heuristic considers three different cases, graphically illustrated in Figure 7, and discussed next.

(1) *Optimal case*. If $T \cap S \ne \emptyset$, then there is a target in the successor set, and it is synthesized. If the entire set T is synthesized this way, then the solution is optimal, since it is impossible to use less than one $\mathcal{A}$-operation for each odd target. Thus, this case is called optimal.
(2) *Heuristic case A*. If $T \cap S = \emptyset$ and $T \cap S^2 \ne \emptyset$, then there is a target at an $\mathcal{A}$-distance of 2 from R. This target is synthesized along with the distance-1 intermediate fundamental.
(3) *Heuristic case B*. If no distance-1 or distance-2 targets are available, then RAG-n synthesizes the target of least complexity using the precomputed optimal SCM table. In this case, three or more constants are synthesized.

RAG-n computes the entire set S and then finds the intersection $S \cap T$ to detect distance-1 targets. Distance-2 targets, on the other hand, are detected using a *heuristic distance test* only. This test is less expensive than a full computation of S^2 but does not detect all distance-2 numbers. We express this in Figure 7 by using a dashed circle for S^2.

The last case (heuristic case B) uses the precomputed SCM decomposition to decompose a target. In general, this does not lead to any subexpression-sharing. With the current methods, the largest available optimal SCM table is restricted to constants of up to 19 bits [Gustafsson et al. 2002], which also limits the applicability of RAG-n.

RAG-n applied to a single constant set will always yield an optimal decomposition if the constant resides in the precomputed optimal SCM lookup table. Constants not in the lookup table are only synthesized at all (optimal or not) if they have complexity 1 or 2, that is, have a distance 1 or 2 from the initial $R = \{1\}$. Since RAG-n uses the lookup table, it can only be considered applicable to target sets with 2 or more constants. Other MCM algorithms, including BHA, BHM, and the proposed new algorithm, can be applied to the SCM problem directly.

3.5. HYBRID GRAPH-BASED ALGORITHMS. The common structure of graph-based algorithms makes it possible to easily mix different algorithms to obtain hybrids. For example, in any given iteration, one can change the heuristic or use a different algorithm to synthesize a target.

For example, RAG-n itself is a hybrid with three components. When its optimal, distance-1 test fails, the algorithm switches to a distance-2 heuristic, and when this also fails, RAG-n reverts to a lookup table to synthesize a fundamental. Similarly, RAG-n can be easily modified to use a CSE-based algorithm, instead of a lookup table, to synthesize the targets not detected by its distance-1 and distance-2 tests.

4. *New Algorithm*

This section first describes limitations of the existing algorithms introduced in Section 3, and then presents in detail the proposed new algorithm. The heuristic in the new algorithm assumes that there is a function that will compute exactly or estimate the $\mathcal{A}$-distance. This is a nontrivial problem and will be addressed separately in Section 5.

4.1. LIMITATIONS OF EXISTING GRAPH-BASED ALGORITHMS. BHA was designed more than a decade ago, when workstation computers had several orders of magnitude less computing power, and thus uses a computationally modest heuristic by today's standards. Although BHM improved on BHA, the heuristic was not changed significantly. So it can be expected that on today's computers we can improve on BHA and BHM by choosing a computationally more complex and thus more precise heuristic.

Although RAG-n performs reasonably well, one main disadvantage is its dependence on a precomputed table of optimal single constant add/subtract/shift decompositions. First, this table takes a time exponential in the number of bits to construct. Second, the best available method to compute this table [Gustafsson et al. 2002] works only up to 19 bits (even though one could also use a good suboptimal table, which was not investigated). Finally, the size of the table is also exponential in the number bits (it must have an entry for every odd constant). For instance, the table for all odd constants up to 32 bits requires $2^{31} > 10^9$ entries. Thus, RAG-n is not applicable to large bitwidths.

Another shortcoming of RAG-n (as will as of BHA and BHM) is that the heuristic does not try to select intermediate fundamentals to *jointly* optimize for all target constants. This often results in suboptimal answers, even in very simple cases. For example, RAG-n applied to the target set $T = \{23, 81\}$ synthesizes a multiplier block that uses four $\mathcal{A}$-operations without any subexpression-sharing (both 23 and

81 are complexity-2 constants). In contrast, Figure 5 shows a better solution with only three operations produced by our algorithm.

4.2. New Algorithm. The main idea behind our new algorithm is to use a better heuristic for synthesizing intermediate fundamentals. Our algorithm is computationally more expensive than BHA, BHM, and RAG-n, since it explores a very large space of possible intermediate vertices. Unlike RAG-n, it does not require a pregenerated optimal SCM lookup table. Thus, our algorithm is storage efficient and in its applicability only limited by the computation time.

Target $\mathcal{A}$-Operation. The $\mathcal{A}$-operation in our algorithm is $\mathcal{A}^{\text{odd}}$ with the same $\mathcal{A}$-configuration constraint as RAG-n, namely, allowing $\mathcal{A}_p(u, v) \leq 2^{b+1}$, where b is the maximal bitwidth of constants in T. When describing the algorithm we will use for simplicity $\mathcal{A}$ to denote $\mathcal{A}^{\text{odd}}$ with the aforementioned constraint.

Outline of the New Algorithm. Our algorithm follows the general structure of Algorithm 1 and is shown in Algorithm 2. The heuristic is split into two parts: the optimal and the heuristic part, similarly to the RAG-n algorithm. The optimal part is practically identical to the optimal part in RAG-n, but in Algorithm 2 we give a more detailed explanation of how to efficiently construct the successor set S. The heuristic part uses S and the $\mathcal{A}$-distance tests and estimators developed in Section 5 to select the new successors s to be added to R. Unlike in RAG-n, our heuristic part only adds a single successor to R at each iteration.

When we talk about a single iteration of the algorithm (e.g., later in its analysis), we refer to the outer loop consisting of steps 5–18.

Next, we discuss Algorithm 2 in greater detail.

Construction of S and the Optimal Part. The optimal part of our algorithm is equivalent to the optimal part of RAG-n. Recall that the optimal part of RAG-n synthesizes at each iteration all distance-1 targets, namely, $S \cap T$.

To avoid computing in each iteration the entire set S which can become rather large, we compute it incrementally. This necessitates an additional set: the worklist W. When a constant is synthesized, it is added to W, first without being accounted for, neither in R nor in S. In steps 9–10 we then perform an incremental update of R and S based on W. The update of R is straightforward (step 9).

$$R_{\text{new}} = R \cup W$$

The update formula for S is derived as follows, using Eqs. 7 and 10:

$$\begin{aligned}
S_{\text{new}} &= \mathcal{A}_*(R_{\text{new}}, R_{\text{new}}) = \mathcal{A}_*(R_{\text{new}}, R \cup W) \\
&= \mathcal{A}_*(R_{\text{new}}, R) \cup \mathcal{A}_*(R_{\text{new}}, W) \\
&= \mathcal{A}_*(R \cup W, R) \cup \mathcal{A}_*(R_{\text{new}}, W) \\
&= \mathcal{A}_*(R, R) \cup \mathcal{A}_*(W, R) \cup \mathcal{A}_*(R_{\text{new}}, W) - W \\
&= S \cup \mathcal{A}_*(R, W) \cup \mathcal{A}_*(R_{\text{new}}, W) - W
\end{aligned}$$

Since $\mathcal{A}_*(R, W) \subset \mathcal{A}_*(R_{\text{new}}, W)$, we get

$$S_{\text{new}} = (S \cup \mathcal{A}_*(R_{\text{new}}, W)) - W, \tag{14}$$

which is step 10 in Algorithm 2.

Algorithm 2 New MCM Algorithm.
Given the target set of constants T. Compute the set $R = \{r_1, \ldots, r_m\}$, with $T \subset R$, as given in Definition 2.2 in Section 2. There is a degree of freedom in choosing the heuristic function $\mathrm{H}(R, S, T)$ for the algorithm. We consider two alternatives $\mathrm{H}_{\mathrm{maxb}}$ and $\mathrm{H}_{\mathrm{cub}}$ discussed in Section 4.3.

```
SynthesizeMultiplierBlock(T)
 1: Right shift elements of $T$ until odd
 2: $R \leftarrow \{1\}$
 3: $W \leftarrow \{1\}$
 4: $S \leftarrow \{1\}$
 5: while $T \neq \emptyset$ do
 6:    {optimal part}
 7:    while $W \neq \emptyset$ do
 8:       {update S and R}
 9:       $R \leftarrow R \cup W$
10:       $S \leftarrow (S \cup \mathcal{A}_*(R, W)) - W$
11:       $W \leftarrow \emptyset$
12:       {if S contains targets, synthesize them}
13:       for $t \in S \cap T$ do
14:          Synthesize($t$)
15:    {heuristic part}
16:    if $T \neq \emptyset$ then
17:       $s \leftarrow \mathrm{H}(R, S, T)$
18:       Synthesize($s$)

Synthesize(s)
 1: $W \leftarrow W + s$
 2: $T \leftarrow T - s$
```

Heuristic Part. When no more targets are found in S, the optimal part of the algorithm cannot synthesize any constants. This means that all targets are more than one $\mathcal{A}$-operation away, and a heuristic function $\mathrm{H}(R, S, T)$ is used to find the next successor to add to R. Adding a successor to R creates new successors, possibly enabling the optimal part to subsequently synthesize new targets.

We have developed two different heuristic functions $\mathrm{H}_{\mathrm{maxb}}$ and $\mathrm{H}_{\mathrm{cub}}$, discussed next.

4.3. Heuristics.

We present two heuristic functions H considered in our algorithm, called *maximum benefit* and *cumulative benefit*. Both heuristics assume that one can compute or estimate the $\mathcal{A}$-distance. This is a nontrivial problem, and the next section is dedicated to $\mathcal{A}$-distance computation/estimation.

Heuristic 1: *Maximum Benefit.* The obvious and natural heuristic that comes to mind, assuming that it is possible to compute or estimate the $\mathcal{A}$-distance, is to pick the successor $s \in S$ closest to the target set. However, it is useful to also take into account the current estimate of the distance between R and T. Thus, to build our heuristic, we first define the *benefit function* $\mathrm{B}(R, s, t)$ to quantify to what extent adding a successor s to the ready set R improves the distance to a fixed, but arbitrary, target t.

$$\mathrm{B}(R, s, t) = \mathrm{dist}(R, t) - \mathrm{dist}(R + s, t) \tag{15}$$

(Recall that we write $R + s$ for $R \cup \{s\}$.) If dist is exact, then B is at most 1. For targets farther away, however, the $\mathcal{A}$-distance can only be estimated, and with these estimated distances the benefit can be larger than 1. Moreover, for remote targets the estimate becomes less accurate (refer to Section 5), but also less important. We take this into account by introducing the *weighted benefit* function

$$\overline{\mathrm{B}}(R, s, t) = 10^{-\mathrm{dist}(R+s,t)}(\mathrm{dist}(R, t) - \mathrm{dist}(R + s, t)), \tag{16}$$

where the weight factor $10^{-\mathrm{dist}(R+s,t)}$ is exponentially decreasing as the distance to t grows. Initially, we always favored closer targets, which is equivalent to having a very large exponent base, but later it was experimentally found that 10 slightly improves the solutions obtained with our second heuristic.

The *maximum benefit heuristic* $\mathrm{H}_{\mathrm{maxb}}(R, S, T)$ used in our algorithm picks the successor $s \in S$ that maximizes the weighted benefit over all targets $t \in T$.

$$\mathrm{H}_{\mathrm{maxb}}(R, S, T) = \arg\max_{s\in S}\left(\max_{t\in T}\overline{\mathrm{B}}(R, s, t)\right) \tag{17}$$

Maximizing $\overline{\mathrm{B}}$ tends to give preference to the successors closest to T, but between two successors that are equally far from T, it prefers the one with the greater benefit value.

Heuristic 2: *Cumulative Benefit*. The key observation about the benefit function is that benefits for different targets t can be summed to enable joint optimization for all targets. This leads to the second and, as it turns out, superior *cumulative benefit heuristic*, formally defined as

$$\mathrm{H}_{\mathrm{cub}}(R, S, T) = \arg\max_{s\in S}\left(\sum_{t\in T}\overline{\mathrm{B}}(R, s, t)\right). \tag{18}$$

The cumulative benefit heuristic adds up the weighted benefits with respect to all targets in T instead of taking the maximum, and thus accomplishes joint optimization for all targets.

Remarks. In a sense, our first heuristic $\mathrm{H}_{\mathrm{maxb}}$ corresponds to a maximum norm $\|\cdot\|_\infty$, and the second heuristic $\mathrm{H}_{\mathrm{cub}}$ to the 1-norm $\|\cdot\|_1$ (of course, a proper norm would require an underlying vector space). We had also considered the equivalent of the 2-norm $\|\cdot\|_2$, but it did not produce results significantly different from $\mathrm{H}_{\mathrm{cub}}$.

4.4. Termination and Distance Function Admissibility. The inner loop (steps 7–14 in Algorithm 2) of the optimal part is guaranteed to terminate, since there is a finite number of targets and at each iteration, either a target is synthesized or the optimal part finishes.

The heuristic part, on the other hand, does not synthesize target constants. Since one constant at-a-time is synthesized, it would have to be a distance-1 target. However, if the test for the optimal part fails, that is, $T \cap S = \emptyset$, it is guaranteed that no distance-1 targets exist. Thus, the heuristic part only synthesizes intermediate vertices, and the algorithm is not guaranteed to terminate unless the heuristic H meets certain conditions.

For $\mathrm{H}_{\mathrm{maxb}}$ and $\mathrm{H}_{\mathrm{cub}}$, the termination is guaranteed if for an arbitrary ready set $R \supseteq \{1\}$, the corresponding successor set S, and any $t \in T$, the $\mathcal{A}$-distance estimation function dist is *admissible*, as defined next.

Definition 4.1 (*Admissible $\mathcal{A}$-Distance Estimation Function*). We call an $\mathcal{A}$-distance estimation function dist admissible if the following holds:

(1) $\operatorname{dist}(R, t)$ is a finite nonnegative integer;
(2) $\operatorname{dist}(R, t) = 0$ iff $t \in R$;
(3) $\operatorname{dist}(R, t) > 0$ iff $t \notin R$;
(4) for an arbitrary set U, $\operatorname{dist}(R \cup U, t) \leq \operatorname{dist}(R, t)$; and
(5) if $t \notin R$, then there exists $s \in S$ such that

$$\mathrm{B}(R, s, t) = \operatorname{dist}(R, t) - \operatorname{dist}(R + s, t) > 0 \quad \text{or} \quad \operatorname{dist}(R + s, t) < \operatorname{dist}(R, t).$$

THEOREM 4.2 (ALGORITHM TERMINATION). *Algorithm* 2 *with the heuristic* $\mathrm{H} = \mathrm{H}_{\mathrm{maxb}}$ *or* $\mathrm{H} = \mathrm{H}_{\mathrm{cub}}$ *terminates if* dist *is admissible.*

PROOF. Consider the sum of estimated distances to all unsynthesized targets $D = \sum_{t \in T} \operatorname{dist}(R, t)$. The admissibility conditions 1–3 in Definition 4.1 imply that D is finite, $D > 0$ for $T \neq \emptyset$, and that $D = 0$ iff $T = \emptyset$, namely, when all targets are synthesized.

Both heuristics choose a successor with positive benefit if it exists.[2] According to condition 5 of Definition 4.1, such a successor s always exists and

$$\operatorname{dist}(R + s, t) < \operatorname{dist}(R, t).$$

Thus, the intermediate fundamental chosen by the heuristic will decrease by at least 1 distance estimate, and since according to condition 4, estimates cannot increase with an addition of new elements to R, the sum D will also decrease.

Since at each iteration D is decreased, eventually it will become 0. Then, according to condition 2, all targets are synthesized and the algorithm terminates. Observe also that D is the maximum number of iterations of the heuristic part. The optimal part decreases D by at least 1 for each synthesized target, and the heuristic part decreases D by at least 1 for each synthesized intermediate constant, thus D is also the maximum number of synthesized constants or $|R|$. □

Obviously, if dist is exact and not an estimate, it is admissible.

4.5. ALGORITHM PROPERTIES. Let $n = |T|$, and let b be the maximal bitwidth of all constants in T. We derive the worst-case sizes of $\mathcal{A}_*$, namely, R (i.e., the worst-case solution) and S, as well as $\mathcal{C}_1$ and $\mathcal{C}_2$. These bounds are necessary for runtime analysis of the $\mathcal{A}$-distance computation in Section 5, hence of the algorithm. Table II summarizes these bounds. We will also identify scenarios in which the algorithm produces an optimal solution. Both worst-case set sizes and optimality hold under the constraint $\mathcal{A}_p(u, v) \leq 2^{b+1}$.

[2] Although $\mathrm{H}_{\mathrm{cub}}$ and $\mathrm{H}_{\mathrm{maxb}}$ use the weighted benefit $\overline{\mathrm{B}}$, it can be easily seen that if $\overline{\mathrm{B}}(R, s, t) > 0$, then also $\mathrm{B}(R, s, t) > 0$.

TABLE II. Worst-Case Set Sizes for the New Algorithm

Set	Worst-Case Size
$\mathcal{A}_*(u, v)$	$O(b)$
$\mathcal{C}_1$	$O(b)$
$\mathcal{C}_2$	$O(b^2)$
T	n
R	$O(nb)$
S	$O(n^2b^3)$

Worst-Case Size of $\mathcal{A}_*$. $\mathcal{A}^{\text{odd}}$ allows only one nonzero left shift l_1 or l_2, and since the fundamental values are limited to 2^{b+1}, the shifts can be between 0 and $b + 1$. The only other parameter that can vary is $s = \{0, 1\}$. Therefore, for a fixed u and v, there are at most $O(4(b + 2)) = O(b)$ possible $\mathcal{A}$-configurations, and hence at most $O(b)$ elements in $\mathcal{A}_*(u, v)$.

Worst Case Solution ($|R|$). As mentioned in the proof of Theorem 4.2, the size of the solution is bounded as

$$|R| \leq D = \sum_{t \in T} \operatorname{dist}(R, t). \tag{19}$$

The $\mathcal{A}$-distance estimation function presented in the next section is bounded by the CSD cost of t. Therefore, for b bit constants (with a CSD cost of $O(b)$) and n targets, the worst-case is

$$|R| = O(nb).$$

Worst-Case Size of S. Recall that $S = \mathcal{A}_*(R, R)$. The number of pairs in R (with repetition) is at most

$$|R| + \binom{|R|}{2} = O(nb) + \binom{O(nb)}{2} = O(n^2b^2).$$

Since for each $r, r' \in R$, $\mathcal{A}_*(r, r')$ contains $O(b)$ elements, we have

$$|S| = O(n^2b^2 \cdot b) = O(n^2b^3).$$

Sizes of $\mathcal{C}_1$ *and* $\mathcal{C}_2$. These principally infinite sets (see Definition 2.3) become finite if we take into account the restriction of $\mathcal{A}_p(u, v) \leq 2^{b+1}$ imposed by the algorithm.

Each odd constant in $\mathcal{C}_1$ is in $\mathcal{A}_p(1, 1)$, since 1 is the only odd constant with complexity zero. Thus,

$$|\mathcal{C}_1| = |\mathcal{A}_*(1, 1)| = O(b).$$

Each constant in $\mathcal{C}_2$ is either in $\mathcal{A}_p(1, c)$ or in $\mathcal{A}_p(c, c)$ for a suitable $c \in \mathcal{C}_1$. Thus,

$$\mathcal{C}_2 = \mathcal{A}_*(\mathcal{C}_1, 1) \cup \bigcup_{c \in \mathcal{C}_1} \mathcal{A}_*(c, c).$$

It is easy to see that $\mathcal{A}_*(c, c) = c \cdot \mathcal{A}_*(1, 1)$, and thus

$$\bigcup_{c \in \mathcal{C}_1} \mathcal{A}_*(c, c) = \bigcup_{c \in \mathcal{C}_1} c \cdot \mathcal{A}_*(1, 1) = \mathcal{C}_1 \mathcal{A}_*(1, 1) = \mathcal{C}_1 \cdot \mathcal{C}_1.$$

We have $\mathcal{C}_2 = \mathcal{A}_*(\mathcal{C}_1, 1) \cup \mathcal{C}_1 \cdot \mathcal{C}_1$ and $|\mathcal{C}_2| = O(b^2)$.

Optimality. In certain situations Algorithm 2 produces an optimal solution, that is, with the minimum possible number of $\mathcal{A}$-operations.

THEOREM 4.3 (SINGLE CONSTANT OPTIMALITY). *If the $\mathcal{A}$-distance function* dist *is exact, then Algorithm* 2 *with either* H_{cub} *or* H_{maxb} *is optimal for a single target* ($n = 1$).

PROOF. Denote the single target with t. Then $|R| \leq \text{dist}(\{1\}, t)$ from Eq. 19. Since the distance function is exact, $\text{dist}(\{1\}, t)$ is the $\mathcal{A}$-complexity of t and the result follows. □

THEOREM 4.4 (MULTIPLE CONSTANT OPTIMALITY). *If the optimal part of Algorithm* 2 *synthesizes the entire set T, then the solution is optimal.*

PROOF. This was already shown for the RAG-n algorithm in Dempster and Macleod [1995], which uses the same optimal part.

If after the first pass of the algorithm T is empty, then R contains all targets, and the solution uses exactly $n = |T|$ $\mathcal{A}$-operations if T has distinct odd constants. It is not possible to use less operations because each unique odd target requires at least one $\mathcal{A}$-operation.

Observe that n is also the lower bound for the number of $\mathcal{A}$-operations for n odd constants. Asymptotically, due to the optimal part of the algorithm, the number of $\mathcal{A}$-operations will approach n when b is fixed. Further discussion can be found in Dempster and Macleod [1995]. □

5. *Computing the $\mathcal{A}$-Distance*

The previous section described the proposed algorithm and the two heuristics H_{cub} and H_{maxb}, both of which are based on a function dist that computes or estimates the $\mathcal{A}$-distance. Our design of this dist function is described in this section.

First, we show that the $\mathcal{A}$-distance computation is an NP-complete problem to motivate the use of estimation. Second, we discuss the special cases in which the $\mathcal{A}$-distance can be computed exactly, and then describe a general method for its estimation. Finally, we prove the admissibility (see Definition 4.1) of the proposed distance function, which guarantees the termination of our algorithm through Theorem 4.2.

THEOREM 5.1 (COMPLEXITY OF COMPUTING $\mathcal{A}$-DISTANCE). *The problem of computing the $\mathcal{A}$-distance under the constraint $\mathcal{A}_p(u, v) \leq 2^{b+1}$ is NP-complete.*

PROOF. We prove this by reducing the NP-complete problem of finding the optimal decomposition for a single constant[3] to the problem of $\mathcal{A}$-distance computation in polynomial time.

[3] The SCM problem with the shift constraint is still NP-complete.

If the $\mathcal{A}$-distance function is exact, then Algorithm 2 is optimal for a single constant (see Theorem 4.3). The heuristic is invoked in the algorithm once per iteration. There are $O(nb) = O(b)$ iterations ($n = 1$ for a single constant) and $O(|S|) = O(n^2b^3) = O(b^3)$ weighted benefits to compute per iteration. Thus the $\mathcal{A}$-distance is computed $O(b^4)$ times. Therefore, the optimal single constant decomposition is reduced to $\mathcal{A}$-distance computation in polynomial time. Hence, $\mathcal{A}$-distance computation is NP-complete. $\square$

Note that computing the $\mathcal{A}$-distance without the shift constraint is a more general problem and thus at least as hard.

We proceed by giving algorithms for computing the *exact* $\mathcal{A}$-distance for distances ≤ 3, and then give a general method that *estimates* the $\mathcal{A}$-distance > 3.

5.1. $\mathcal{A}$-EQUATIONS AND EXACT $\mathcal{A}$-DISTANCE TESTS. The algorithm for finding the exact value of dist(R, t) is based on testing specific distances d for feasibility. First, all possible $\mathcal{A}$-graph topologies that synthesize t using exactly d $\mathcal{A}$-operations are enumerated. Then, these topologies are converted to so-called $\mathcal{A}$-*equations* which relate values at the input, output, and intermediate nodes of the topology. If it is determined that the equation has a solution, then the $\mathcal{A}$-distance is $\leq d$. If we perform these tests in order of increasing distance d, the exact $\mathcal{A}$-distance can be determined. Since the number of graph topologies for a given distance grows quickly [Gustafsson et al. 2002], this approach is feasible only for very small values of d. We consider $d = 1, 2, 3$, and only estimate the large distances.

Before we start, we list to follow a few useful properties of the $\mathcal{A}$-operation $\mathcal{A} = \mathcal{A}^{\text{odd}}$ including constraints (as defined in Section 4.2) used in our algorithm. For other choices of the $\mathcal{A}$-operation, the properties may not hold.

LEMMA 5.2. *If $w = \mathcal{A}_p(u, v)$, then there exists an $\mathcal{A}$-configuration p' such that $u = \mathcal{A}_{p'}(w, v)$.*

PROOF. Using the definition of $\mathcal{A}_p$.

$$w = |2^{l_1}u + (-1)^s 2^{l_2} v|2^{-r}, \quad p = (l_1, l_2, s, r)$$

Solving for u, we obtain

$$\begin{aligned} u &= |2^r w + (-1)^{s'} 2^{l_2} v|2^{-l_1} = \mathcal{A}_{p'}(w, v), \\ p' &= (r, l_2, s', l_1) \text{ for a suitable } s'. \end{aligned}$$

The value of s' is 1 if $s = 0$, and either 0 or 1 if $s = 1$. $\square$

LEMMA 5.3. *If $w = \mathcal{A}_p(u, v)$, then there exists an $\mathcal{A}$-configuration p' such that $w = \mathcal{A}_{p'}(v, u)$.*

PROOF. Obviously, the $\mathcal{A}$-operation is symmetric, and it suffices to switch the left shifts to obtain p'. $\square$

The following two corollaries follow immediately, using the definition of $\mathcal{A}_*$ (Definition 2.5).

COROLLARY 5.4. *For any u and v*

$$\mathcal{A}_*(u, v) = \mathcal{A}_*(v, u).$$

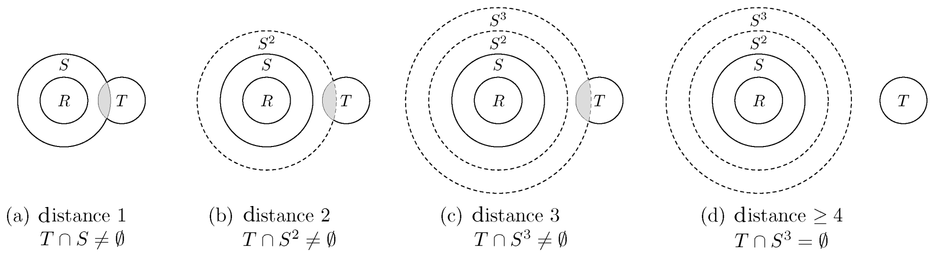
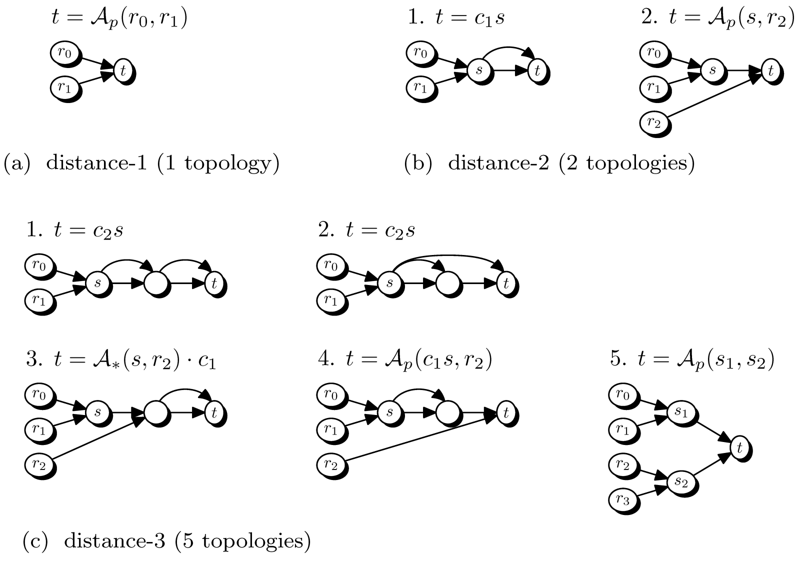


FIG. 8. Special distance cases handled by the heuristic. Solid circles denote available sets, and dashed circles denote the sets that are not computed.



$t = \mathcal{A}_p(r_0, r_1)$
(a) distance-1 (1 topology)
1. $t = c_1 s$
2. $t = \mathcal{A}_p(s, r_2)$
(b) distance-2 (2 topologies)
1. $t = c_2 s$
2. $t = c_2 s$
3. $t = \mathcal{A}_*(s, r_2) \cdot c_1$
4. $t = \mathcal{A}_p(c_1 s, r_2)$
5. $t = \mathcal{A}_p(s_1, s_2)$
(c) distance-3 (5 topologies)

FIG. 9. Graph topologies for exact distance tests.

COROLLARY 5.5. *If* $w \in \mathcal{A}_*(u, v)$*, then*

$$u \in \mathcal{A}_*(w, v) \text{ and } u \in \mathcal{A}_*(v, w),$$
$$v \in \mathcal{A}_*(u, w) \text{ and } v \in \mathcal{A}_*(w, u).$$

We use these properties of $\mathcal{A}$ for solving $\mathcal{A}$-equations that arise in designing the distance tests.

To test for a specific $\mathcal{A}$-distance $\operatorname{dist}(R, t) = d$, we need all graph topologies with d nodes, one or more inputs, and a single output (corresponding to t). We construct these topologies from the cost (1)–(3) SCM topologies from Figure 5 in Gustafsson et al. [2002] by splitting the single input node into multiple input nodes.

The tests for $\operatorname{dist}(R, t)$ proceed by assigning a value $r_i \in R$ to each input node, assigning t to the output node, and solving for all possible values at the first successor node. If any of those values do exist in S, then the test succeeds. We consider the distances $d = 1, 2, 3$ separately next.

5.2. DISTANCE-1 TESTS.

Figure 8(a) displays the case of a distance-1 target, $\operatorname{dist}(R, t) = 1$. For distance-1 there is only one possible topology, shown in Figure 9(a).

Although distance-1 targets can be detected with an $\mathcal{A}$-equation-based test, it is not necessary. The optimal part constructs the entire S (the set of distance-1 constants) and immediately synthesizes all distance-1 targets, namely, the intersection $S \cap T$.

Assuming that S is sorted, the runtime of the test is dominated by set intersection, which can be done in $O(|T| \log |S|) = O(n \log(n^2 b^3)) = O(n \log(nb))$ time.

5.3. Distance-2 Tests. Figure 8(b) shows the scenario where distance-2 targets exist. A distance-2 constant can be realized with two possible topologies, shown in Figure 9(b). Next we construct an $\mathcal{A}$-equation for each of the cases.

Case 1. The subgraph from s to t has a single input and a single output, and hence is a multiplier block with one $\mathcal{A}$-operation which multiplies by a complexity-1 constant. Thus t can be expressed as $t = c_1 s$, where $c_1 \in \mathcal{C}_1$. Alternatively, $s = \frac{t}{c_1}$, which has a solution iff

$$\frac{t}{\mathcal{C}_1} \cap S \neq \emptyset.$$

Since $|\mathcal{C}_1| = O(b)$, and intersection with S (assumed to be sorted) gives another factor of $O(\log |S|) = O(\log(nb))$, this test requires $O(b \log(nb))$ time.

Case 2. Given $t = \mathcal{A}_p(s, r_2)$, we obtain $s = \mathcal{A}_{p'}(t, r_2)$ using Lemma 5.2, which has a solution iff

$$\mathcal{A}_*(t, R) \cap S \neq \emptyset.$$

$\mathcal{A}_*$ has $O(b)$ elements (and takes the same amount of time to compute) for each pair of inputs. Thus, for each $O(nb)$ element in R we have to perform $O(b)$ operations. Intersection with (sorted) S gives another factor of $O(\log |S|) = O(\log(nb))$, and the total time for this test is $O(nb^2 \log(nb))$.

5.4. Distance-3 Tests. Figure 8(c) shows the scenario where distance-3 targets exist. A distance-3 constant can be realized with 5 possible topologies, shown in Figure 9(c).

Gustafsson et al. [2002] present all graphs in so-called *reduced* form, which allows vertices to have an in-degree larger than 2. For topologies 2, 3, 4, and 5, we have chosen to split these vertices into binary vertices. It is shown by the authors that different splittings have equivalent possible outputs at the output vertex. We have chosen splittings that minimize the runtime of the distance test.

Cases 1 *and* 2. In both of these cases, the subgraph from s to t has a single input and a single output, hence is a multiplier block which multiplies by a complexity-2 constant. Cases 1 and 2 consider different complexity-2 constants, but both can be covered if the target value is written as $t = c_2 s$, where $c_2 \in \mathcal{C}_2$. Alternatively, $s = \frac{t}{c_2}$, which has a solution iff

$$\frac{t}{\mathcal{C}_2} \cap S \neq \emptyset.$$

There are $O(b^2)$ complexity-2 constants (see Table II). Test membership in S can be done in $O(\log(nb))$ operations, so this test requires $O(b^2 \log(nb))$ time.

TABLE III. SET INTERSECTIONS COMPUTED FOR EXACT DISTANCE TESTS AND COMPUTATION TIME PER TARGET

Distance	Case	Formula	Time per Target
1	–	$t \cap S$	$O(\log(nb))$
2	1	$\mathcal{A}_*(R, t) \cap S$	$O(nb^2 \log(nb))$
	2	$\frac{t}{\mathcal{C}_1} \cap S$	$O(b \log(nb))$
3	1,2	$\frac{t}{\mathcal{C}_2} \cap S$	$O(b^2 \log(nb))$
	3	$\mathcal{A}_*\left(\frac{t}{\mathcal{C}_1}, R\right) \cap S$	$O(nb^3 \log(nb))$
	4	$\frac{\mathcal{A}_*(R,t)}{\mathcal{C}_1} \cap S$	$O(nb^3 \log(nb))$
	5	$\mathcal{A}_*(S, t) \cap S$	$O(n^2b^4 \log(nb))$

Case 3. The $\mathcal{A}$-equation can be rewritten as $\mathcal{A}_p(s, r_2) = \frac{t}{c_1}$. Using Lemma 5.2 we get $\mathcal{A}_{p'}(\frac{t}{c_1}, r_2) = s$, which has a solution iff

$$\mathcal{A}_*\left(\frac{t}{\mathcal{C}_1}, R\right) \cap S \neq \emptyset.$$

Since $|\mathcal{C}_1| = O(b)$, the time for this test is $O(b \cdot nb \cdot b \cdot \log(nb)) = O(nb^3 \log(nb))$.

Case 4. Given $t = \mathcal{A}_p(c_1 s, r_2)$ and using Lemma 5.2, we obtain $c_1 s = \mathcal{A}_{p'}(t, r_2)$, or, alternatively, $s = \frac{\mathcal{A}_{p'}(t,r_2)}{c_1}$, which has a solution iff

$$\frac{\mathcal{A}_*(R, t)}{\mathcal{C}_1} \cap S \neq \emptyset.$$

This test is similar to the second distance-2 test, but for each element of $\mathcal{A}_*(R, t)$ it has to go through all elements of $\mathcal{C}_1$. Since there are $O(b)$ constants in $\mathcal{C}_1$, the required time is $O(b \cdot nb^2 \log(nb)) = O(nb^3 \log(nb))$.

Case 5. Given $t = \mathcal{A}_p(s_1, s_2)$ and using Lemma 5.2, we obtain $s_2 = \mathcal{A}_{p'}(s_1, t)$, which has a solution iff

$$\mathcal{A}_*(S, t) \cap S \neq \emptyset.$$

For each $s_1 \in S$, we have to perform $O(b \log(nb))$ operations. Thus the time for this test is $O(|S| \cdot b \log(nb)) = O(n^2b^4 \log(nb))$.

5.5. SUMMARY OF EXACT TESTS. Table III shows the set intersections that need to be computed for each of the exact $\mathcal{A}$-distance tests, and the corresponding asymptotic runtime per each tested target. In all cases, the runtime per target is equal to the worst-case set size times $\log(nb)$ overhead for the intersection with S, assuming that S is sorted.

The preceding tests yield the exact value of $\text{dist}(R, t)$. Adding a single element to R can decrease the $\mathcal{A}$-distance by at most 1; this implies that $\text{dist}(R + s, t)$, which is needed for calculating $\overline{\text{B}}(R, s, t)$, does not have to be computed at all.

All given tests compute the set, call it X, of possible values at the successor node s, and then check whether $X \cap S \neq \emptyset$. For all $s \in X \cap S$ it holds that $\text{dist}(R + s, t) = \text{dist}(R, t) - 1$.

The algorithm requires an admissible distance function (Section 4.4). For $\mathcal{A}$-distances up to 3, we use the exact distance function which is admissible. For larger distances we estimate the $\mathcal{A}$-distance, explained next.

5.6. ESTIMATION (FOR DISTANCE-4 AND HIGHER). Figure 8(d) shows the scenario where no targets of distance 3 or lower exist. We do not use an exact distance computation in this case. However, the exact test is still feasible, and could be designed with the method shown before from the 15 possible distance-4 topologies.

Our approach estimates distances of 4 or higher using several estimators, each of which overestimates the exact distance. The final estimated distance is then the minimum of these overestimates.

Recall that for the weighted benefit $\overline{\mathrm{B}}(R, s, t)$, we need both $\mathrm{dist}(R, t)$ and $\mathrm{dist}(R + s, t)$. For the exact tests described previously, $\mathrm{dist}(R, t)$ is computed, and $\mathrm{dist}(R + s, t)$ for each successor is obtained as a side-effect. For estimation, the converse is true. The distances $\mathrm{dist}(R, t)$ are obtained from $\mathrm{dist}(R + s, t)$ and then cached, as explained in the following.

Initially, for targets t which are not covered by exact distance tests, the cached value of $\mathrm{dist}(R, t)$ is set to the largest possible distance value. Each time the benefit function has to be computed and the exact tests do not apply, $\mathrm{dist}(R, t)$ is obtained from the cache, and $\mathrm{dist}(R + s, t)$ is estimated using the method given here. If a particular successor s is chosen to be synthesized, the computed value of $\mathrm{dist}(R + s, t)$ replaces $\mathrm{dist}(R, t)$ in the cache.

For exact tests, both the $\mathrm{H}_{\mathrm{cub}}$ and $\mathrm{H}_{\mathrm{maxb}}$ heuristics only need a single $\mathrm{dist}(R, t)$ computation for each target t. Estimation, on the other hand, is much more expensive because it requires $\mathrm{dist}(R + s, t)$ for each of the $O(|S|) = O(n^2 b^3)$ successors s for each target t.

Estimating $\mathrm{dist}(R + s, t)$. To estimate the $\mathcal{A}$-distance, we try to find an answer to the following two questions: What constant z do we need to reduce the distance to a target? And how expensive is z?

To answer the first question, we construct solutions to the problem of synthesizing t using two and three $\mathcal{A}$-operations. These partial solutions are graph topologies, as in the exact test cases in Figure 9(b) and 9(c), but have one of the inputs designated as an unknown and unsynthesized constant z. Since besides a successor node s, we also need z, the graphs must have at least 3 inputs. Note that there are exactly 4 topologies with 3 or more inputs in Figure 9: distance-2 topology 2, and distance-3 topologies 3, 4, and 5. These are the graph topologies used for distance estimation, and Figure 10 repeats them showing also the designated unknown input z. Using these topologies and Lemmas 5.2 and 5.3 (as we did for exact distance tests), we can compute the set of all possible values of z.

To answer the second question we use a crude estimate for $\mathrm{dist}(R + s, z)$, called a single constant *auxiliary cost measure* $\mathrm{Est}(z)$, as the cost.

For a given partial topology and a given value of z, an overestimated distance can be obtained as

$$\begin{aligned}\mathrm{dist}(R + s, t) &\le \mathrm{dist}(R + s, z) + \mathrm{dist}(R + s + z, t)\\ &= \mathrm{Est}(z) + \#\mathrm{ops} - 1,\end{aligned}$$

where #ops is the number of nodes (i.e., number of $\mathcal{A}$-operations) in the topology, and, since s is assumed to be available, 1 is subtracted.

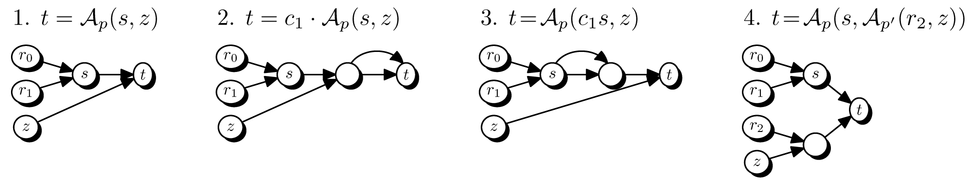

1. $t = \mathcal{A}_p(s, z)$
2. $t = c_1 \cdot \mathcal{A}_p(s, z)$
3. $t{=}\mathcal{A}_p(c_1 s, z)$
4. $t{=}\mathcal{A}_p(s, \mathcal{A}_{p'}(r_2, z))$
r_0
r_1
r_2
s
t
z

FIG. 10. Partial graph topologies for dist$(R + s, t)$ estimation. z denotes the unsynthesized part. Estimation proceeds by determining all possible values of z, finding the "cheapest" (we use CSD cost), and adding the cost of the least expensive element to the number of operations in the topology minus 1 (since s is assumed to be available).

However, since the value of z is unknown, we compute the set Z of all possible values of z for a given partial topology as well as the values of s and t, and then use the "cheapest" (with respect to Est) value, denoted Est(Z).

$$\begin{aligned} \text{dist}(R + s, t) &\leq \min_{z \in Z} \text{Est}(z) + \#\text{ops} - 1 \\ &= \text{Est}(Z) + \#\text{ops} - 1 \end{aligned} \tag{20}$$

There is a degree of freedom in choosing the auxiliary cost measure Est. However, first we must ensure that the distance function is admissible, and then that it is computationally efficient. For example, the constant complexity cannot be used, because the resulting distance function will not be admissible. Among the possible choices are the number of nonzero CSD bits (CSD cost) of the constant, or the number of nonzero binary bits of the constant. We have chosen the CSD cost as an auxiliary estimator.

$$\text{Est}(z) = \text{CSD-Cost}(z), \tag{21}$$

$$\text{Est}(Z) = \min_{z \in Z} \text{Est}(z) = \min_{z \in Z} \text{CSD-Cost}(z) \tag{22}$$

Computing the CSD cost takes $O(b)$ time, and we show later that the resulting $\mathcal{A}$-distance estimation function is admissible.

Using Eq. 20 for distance estimation and Lemma 5.2 to compute Z using computations similar to the exact distance test, we obtain the following $\mathcal{A}$-distance overestimates for each case in Figure 10:

Case 1. $\text{dist}(R + s, t) \leq 1 + \text{Est}(\mathcal{A}_*(s, t)) = E_1$.

Case 2. $\text{dist}(R + s, t) \leq 2 + \text{Est}(\mathcal{A}_*(s, \frac{t}{\mathcal{C}_1})) = E_2$.

Case 3. $\text{dist}(R + s, t) \leq 2 + \text{Est}(\mathcal{A}_*(\mathcal{C}_1 s, t)) = E_3$.

Case 4. $\text{dist}(R + s, t) \leq 2 + \text{Est}(\mathcal{A}_*(R, \mathcal{A}_*(s, t))) = E_4$.

The estimates E provide an upper bound and thus can be larger than dist(R, t). When all estimates are larger than dist(R, t), the benefit B(R, s, t) must be 0 (and not negative), and therefore, as the final value for the estimate of dist$(R + s, t)$ we take the minimum of the four overestimates and dist(R, t).

$$\text{dist}(R + s, t) \simeq \min(\text{dist}(R, t), E_1, E_2, E_3, E_4) \tag{23}$$

Here and further on we will use $\simeq$ to denote that the righthand size is an estimate.

TABLE IV. SETS COMPUTED FOR DISTANCE ESTIMATION AND ESTIMATION RUNTIME PER EACH SUCCESSOR-TARGET PAIR

Case	Set Z	Set Size	Est(Z) Runtime
1	$\mathcal{A}_*(s, t)$	$O(b)$	$O(b^2)$
2	$\mathcal{A}_*(s, \frac{t}{\mathcal{C}_1})$	$O(b^2)$	$O(b^3)$
3	$\mathcal{A}_*(\mathcal{C}_1 s, t)$	$O(b^2)$	$O(b^3)$
4	$\mathcal{A}_*(R, \mathcal{A}_*(s, t))$	$O(nb^3)$	$O(nb^4)$

Table IV shows for each case the set Z (of z values) to compute, the worst-case size of the set, and the runtime for computing Est(Z) ("Est(Z) runtime") for each successor-target pair, assuming that computing Est(z) takes $O(b)$ time.

In our actual implementation of Algorithm 2, the fourth estimator is not used, since it is the most expensive and only insignificantly improves the results. Further, the sets required for the first three estimators do not change between iterations, thus Est(Z) can be computed once for each successor-target pair.

5.7. ESTIMATION ADMISSIBILITY. We already mentioned before that the exact distance function is admissible.

The distance estimate using Eq. 23 obviously satisfies the admissibility conditions 1–4 in Definition 4.1. However, it is not as obvious that condition 5 in this definition is satisfied. We prove this in the following theorem.

Recall that the estimate for dist(R, t) is obtained from that for dist$(R + s, t)$ when s is chosen to be synthesized. Initially, all cached estimates are assumed to be infinite. Therefore, at the first iteration we are guaranteed that the obtained estimate d satisfies $d <$ dist(R, t). The next theorem proves that this will continue to be the case for following iterations, so long as the given estimators are used.

THEOREM 5.6. *Let R, S, T be the ready set, successor set, and target set, respectively. Let $s \in S$, and $t \in T$, and let* dist$(R+s, t) \simeq d$ *be the distance estimate obtained from Eq. 23. Further, assume that $d <$* dist(R, t) *and thus* B$(R, s, t) > 0$. *Then, at the next iteration, there exists $\bar{s} \in S_{\text{new}}$ (the new successor set of $R_{\text{new}} = R + s$) such that* dist$(R + s + \bar{s}, t) = d - 1$, *namely,* B$(R + s, \bar{s}, t) > 0$.

PROOF. There are four cases to consider: $d = E_1$, $d = E_2$, $d = E_3$, and $d = E_4$. We will show only the case $d = E_1$; the proofs for the other cases are analogous.

According to the Case 1 estimator $\mathcal{A}$-equation,

$$t \in \mathcal{A}_*(s, z).$$

Since $d \geq 3$, $Est(z) \geq 2$, that is, the CSD cost of z is at least 2. If one nonzero CSD bit is removed from z, we get a new constant $\bar{z}$ with a CSD cost that is reduced by 1, and z can be written as

$$z = \mathcal{A}_p(1, \bar{z}) \quad \text{so that } \operatorname{Est}(\bar{z}) = \operatorname{Est}(z) - 1.$$

If this expression for z is substituted into the original $\mathcal{A}$-equation, we obtain

$$t \in \mathcal{A}_*(s, \mathcal{A}_p(1, \bar{z})).$$

Using the definition of $\mathcal{A}^{\text{odd}}$ it can be easily shown that there exists an

$\mathcal{A}$-configuration p' such that

$$t \in \mathcal{A}_*(\mathcal{A}_{p'}(s, 1), \overline{z}).$$

However, $\mathcal{A}_{p'}(s, 1) \subset S_{\text{new}}$, hence

$$t \in \mathcal{A}_*(S_{\text{new}}, \overline{z}).$$

Therefore there exists $\overline{s} \in S_{\text{new}}$ with $t \in \mathcal{A}_*(\overline{s}, \overline{z})$, and using the Case 1 overestimate $\text{dist}((R + s) + \overline{s}, t) \leq 1 + \text{Est}(\overline{z}) = 1 + \text{Est}(z) - 1 = d - 1$. □

In the proof, we assumed that an estimate d was obtained using E_1, and found that the value of E_1 again will necessarily decrease. For the other estimates E_2–E_4 of d, however, at the next iteration only a different estimate might decrease.

We have experimented with more expensive estimators. Often, the proof does not go through without an additional and even more expensive estimator that guarantees admissibility.

6. *Runtime Analysis and Experimental Evaluation*

In this section we analyze the runtime of the new algorithm (i.e., Algorithm 2) and its performance in terms of the average number of $\mathcal{A}$-operations (add/subtract operations) in the synthesized solutions. We compare to the best-performing algorithms from the literature. We also provide some measured sample runtimes of the new algorithm. As before, we will use b to denote the constant bitwidth, and $n = |T|$ to denote the number of constants in the target set.

6.1. ALGORITHMS EVALUATED. We provide some details on the actual implementations of the algorithms used in our benchmarks. In some cases, we reimplemented and improved the published algorithms for fair comparison.

New Algorithm with Heuristic H_{cub}. We have implemented our Algorithm 2 in C++, and tried both the H_{cub} and H_{maxb} heuristics from Section 4.3. Both have the same computation cost, but H_{cub} consistently found better solutions. Thus, we present the results of H_{cub} only, and for convenience the algorithm is simply abbreviated as H_{cub}.

Optimal SCM. This method performs exhaustive search over all possible graph topologies to find optimal single constant decompositions. It was originally described in Dempster and Macleod [1994], and later improved in Gustafsson et al. [2002] to handle constants up to 19 bits. We have reimplemented the algorithm in C++ and cross-checked the generated constant complexities with the authors.

RAG-n. Described in Dempster and Macleod [1995] and discussed in Section 3, RAG-n is currently the best published MCM algorithm of which we are aware. The authors have kindly provided us with their MATLAB implementation, which, however, only handles constants up to 12 bits. For a fair comparison, we have reimplemented the algorithm in C++, and generated the lookup table up to 19 bits (using Gustafsson et al. [2002]). We, further improved the algorithms by inserting our complete $\mathcal{A}$-distance test for distance 2, in which case the original implementation used a heuristic only. All RAG-n results shown were produced using this improved version.

TABLE V. ASYMPTOTIC RUNTIME SUMMARY

Algorithm	Runtime	b	$n = \|T\|$
Optimal [Gustafsson et al. 2002]	$\Omega(2^b)$	≤ 19	1
RAG-n [Dempster and Macleod 1995]	$O(n^2 b^3 \log(nb))$	≤ 19	≥ 2
BHM [Dempster and Macleod 1995]	$O(n^3 b^4)$	any	any
Lefèvre [Lefèvre 2001]	$O(n^3 b^3)$	any	any
H_{cub}	$O(n^4 b^5 \log(nb) + n^3 b^6)$	any	any
H_{cub} (distance-2 tests only)	$O(n^3 b^5)$	any	any

BHM. Also described in Dempster and Macleod [1995], BHM is an improved version of BHA (i.e., the "add/subtract/shift" algorithm from Bull and Horrocks [1991]). Both BHA and BHM are described in Section 3. We have implemented this algorithm in C++ using the BHA pseudocode given in Bull and Horrocks [1991] and the BHM improvements from Dempster and Macleod [1995].

Lefèvre. Described in Lefèvre [2001], this is one of the newer common subexpression-elimination-based MCM algorithms. We did not discuss the details of the algorithm in this article, since it is not graph-based. The algorithm uses more sophisticated methods for identifying common subexpressions, but otherwise is similar to Pasko et al. [1999]. The author has kindly provided us with his implementation in Perl.

Note that we implemented every algorithm except the CSE-based one from Lefèvre. This was facilitated by the general framework presented in this article, which enables considerable code reuse. All four algorithms (i.e., ours, optimal SCM, RAG-n, and BHM) require only about 1900 lines of C++ code. Our package implementing these algorithms is available at www.spiral.net.

6.2. ASYMPTOTIC RUNTIME ANALYSIS. Table V summarizes asymptotic bounds for the worst-case runtimes of the different MCM algorithms. Next we describe how they were obtained.

H_{cub}. To derive the worst-case runtime of the new algorithm, we use the worst-case bounds from Sections 4.5 and 5.

The algorithm executes three conceptually separate parts at each iteration: the incremental successor set S construction, the optimal part, and the heuristic part. To follow we show the runtime of each of these steps, and then compute the total runtime using the $O(nb)$ bound for the number of iterations.

—*Successor set construction.* The computation of S is done in increments and distributed across iterations. The total number of successors when the algorithm terminates is $O(n^2 b^3)$. We assume that S is kept sorted in order to do quick set intersections, thus the total runtime is $O(n^2 b^3 \log |S|) = O(n^2 b^3 \log(nb))$.

—*Optimal part.* The only overhead of the optimal part over the computation of S is checking for targets in the new successors, that is, the computation of $S' \cap T$. For n targets, the runtime per iteration is $O(|T| \log |S|) = O(n \log(n^2 b^3)) = O(n \log(nb))$.

—*Heuristic part.* At each iteration we perform a series of exact tests and, if applicable, distance estimators. Table III shows the per-target runtime for the exact tests which have to be evaluated at every iteration. Table IV shows the distance estimator runtime per successor-target pair which do not have to be recomputed between iterations.

TABLE VI. RUNTIME BREAKDOWN FOR THE NEW ALGORITHM

	Per-Iteration Runtime	Total Runtime
S computation	–	$O(n^2b^3\log(nb))$
Optimal part	$O(n\log(nb))$	$O(n^2b\log(nb))$
Heuristic part, exact distance tests	$O(n^3b^4\log(nb))$	$O(n^4b^5\log(nb))$
Heuristic part, distance estimation	–	$O(n^3b^6)$
Total	–	$\boldsymbol{O(n^4b^5\log(nb)+n^3b^6)}$

TABLE VII. RUNTIME BREAKDOWN FOR THE NEW ALGORITHM WITH DISTANCE-2 TESTS ONLY

	Per-Iteration Runtime	Total Runtime
S computation	–	$O(n^2b^3\log(nb))$
Optimal part	$O(n\log(nb))$	$O(n^2b\log(nb))$
Heuristic part, exact distance tests	$O(n^2b^2\log(nb))$	$O(n^3b^3\log(nb))$
Heuristic part, distance estimation	–	$O(n^3b^5)$
Total	–	$\boldsymbol{O(n^3b^5)}$

The most expensive exact distance test is Case 5 of Table III with the runtime of $O(n^2b^4\log(nb))$ per target, in which for n targets we obtain the per-iteration runtime of $O(n^3b^4\log(nb))$ and since the worst-case number of iterations is $O(nb)$, the total runtime is $O(n^4b^5\log(nb))$.

The most expensive distance estimators that we use are Cases 2 and 3 of Table IV with the runtime of $O(b^3)$ per each successor-target pair. As discussed earlier, we do not use the Case 4 estimator. The sets Z computed for Cases 1–3 estimators do not change between iterations, therefore, the estimator values have to be computed only once per each successor-target pair. There are $O(|T|\cdot|S|)=O(n^3b^3)$ such pairs, thus the total runtime is $O(n^3b^3\cdot b^3)=O(n^3b^6)$.

Table VI summarizes the per-iteration and total runtimes for the new algorithm. As can be seen from the table, the total runtime of the algorithm is dominated by the exact distance tests and distance estimators in the heuristic part and is $O(n^4b^5\log(nb)+n^3b^6)$.

If only distance-2 tests and estimators are used, the most expensive distance test is distance-2 of Case 1 in Table III with the runtime of $O(nb^2\log(nb))$ per target, which for n targets yields the runtime of $O(n^2b^2\log(nb))$ per iteration.

The only available distance-2-based estimator is that of Case 1 of Table IV with runtime of $O(b^2)$ per successor-target pair, yielding the total runtime of $O(n^3b^3\cdot b^2)=O(n^3b^5)$.

Table VII summarizes the per-iteration and total runtimes for the new algorithm with distance-2 tests only. Note that the total runtime is now dominated by the distance estimation and decreased to $O(n^3b^5)$.

Optimal SCM. The optimal method for SCM performs an exhaustive search over all possible decompositions, and thus has the highest runtime among all methods. The method has to look at every constant of given bitwidth, so the runtime is $\Omega(2^b)$. Exact analysis was not provided in the original article. The size of the generated lookup table is $O(2^b)$.

RAG-n. The authors of RAG-n did not provide a runtime analysis for the algorithm, so we perform the analysis here.

Since at each iteration at least one target is synthesized, the total number of iterations in the worst-case is $O(|T|) = O(n)$. The optimal part is equivalent to H_{cub} with a runtime of $O(n^2b^3 \log(nb))$. When the optimal part is not applicable, the same distance-2 test as in our algorithm is invoked (which is our improvement of the original, as stated in the beginning of this section). According to Section 5.1, the per-iteration per-target runtime is $O(nb^2 \log(nb))$. At each iteration, the number of targets is decreased by at least 1, thus the total runtime is

$$O((n + (n-1) + (n-2) + \cdots) \cdot b^2 \log(nb)) = O(n^2b^2 \log(nb)).$$

When the distance-2 test fails, RAG-n uses an $O(1)$-table lookup. Thus, the total runtime is dominated by successor set construction in the optimal part and is $O(n^2b^3 \log(nb))$. This differs from the original RAG-n article [Dempster and Macleod 1995] in which the authors observed the heuristic part to be slower. The reason is that our distance-2 test is more efficient.

BHM. For BHM, the runtime analysis was also not available. We do it this, first obtaining a bound on the number of iterations.

The error ϵ starts as a b to follow bit number equal to one of the targets, and at each iteration is reduced by at least a factor of four, namely, by 2 bits, until it reaches 0 and a new target is selected. This gives a total of $O(\frac{b}{2} \cdot |T|) = O(nb)$ iterations.

At each iteration, the heuristic makes a pass through the entire set S according to Eq. (13). Thus, the total runtime is $O(|S| \cdot b \cdot nb) = O(n^2b^3 \cdot nb) = O(n^3b^4)$. It can be verified that the bound $|S| = O(n^2b^3)$ derived for our algorithm still holds (BHM uses an almost equivalent $\mathcal{A}$-operation constraint, and $|R|$ is still $O(nb)$).

BHM has a higher runtime than RAG-n, while Dempster and Macleod [1995] stated otherwise, and created a hybrid RAG-n + BHM algorithm to make it faster. The authors' implementation of the distance-2 test was suboptimal, and with the $\mathcal{A}$-distance computation presented in this framework, RAG-n runs much faster.

Lefèvre. The asymptotic runtime of Lefèvre's algorithm was provided to us by the author; he also noted that the average runtime is lower.

6.3. Experimental Evaluation. To evaluate the performance of different algorithms, we ran a series of experiments on a large random sample of uniformly distributed target sets and measured the average number of adds/subtracts in MCM decompositions.

In the first experiment, we fix n (i.e., the number of constants in the target set) and vary b (i.e., the bitwidth of constants). In the second experiment, we fix b and vary n. In both experiments, we consider BHM, Lefèvre, RAG-n, and H_{cub}. In the third experiment, we investigate the improvement of our algorithm over RAG-n. The fourth experiment shows the gain obtained by using distance-3 tests compared to only distance-2 tests in the algorithm. Finally, we show the actual runtime of our algorithm on a 3.4 GHz Pentium 4 workstation.

Fixed Number of Constants. This experiment investigates the effect of changing the constant bitwidth b on the number of $\mathcal{A}$-operations for $n = 1, 2, 10$, and 20. Figure 11 shows the average number of operations (y axis) versus b (x axis) for 100 uniformly drawn random target sets of size n.

For a single constant, RAG-n uses an optimal SCM decomposition, and therefore is not shown separately. H_{cub} is within 4% of an optimal decomposition at

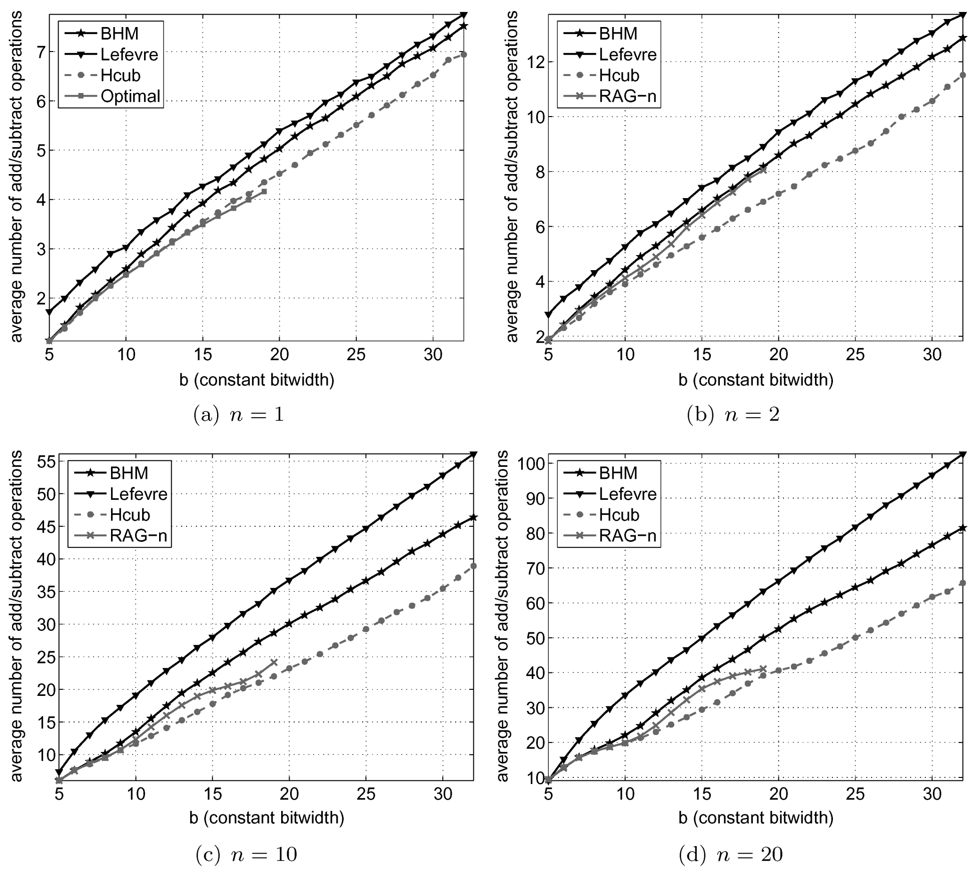


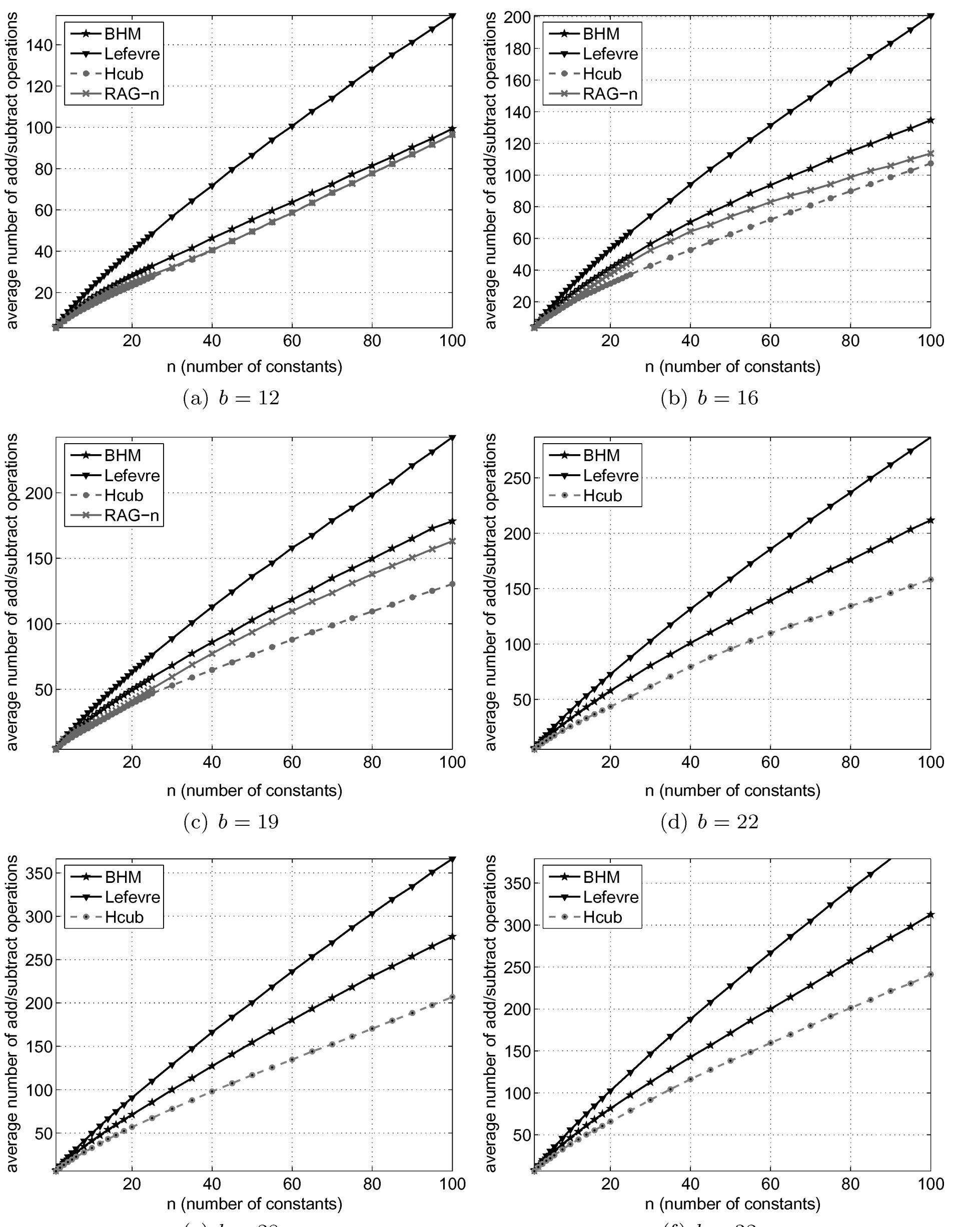
FIG. 11. Average number of $\mathcal{A}$-operations (adds/subtracts) vs. constant bitwidth b for a fixed number of constants n. The average is taken over 100 uniformly drawn random target sets.

$b = 19$ bits, which is the largest bitwidth handled by the optimal algorithm. As the number of bits increases, H_{cub} approaches BHM slightly because the $\mathcal{A}$-distance estimation used in H_{cub} gives increasingly less accurate results. This indicates that for large bitwidths, a hybrid H_{cub} + BHM-based method might be beneficial. Refer to Section 3.5 for a discussion of how hybrid algorithms can be implemented.

As the number of constants increases (i.e., $n = 2, 10, 20$), the performance deterioration effect of $\mathcal{A}$-distance estimation at large bitwidths is delayed, since the joint optimization nature of H_{cub} outweighs the drawbacks of less accurate $\mathcal{A}$-distance estimations.

In particular, H_{cub} performs well for two constants. For bitwidths larger than 14 bits, it requires 10–15% fewer adds/subtracts than RAG-n, which is the best of all other algorithms. Beyond 19 bits, where RAG-n is not applicable, H_{cub} uses up to 17% fewer operations. Sets with 2 constants are an important case in linear signal transforms such as the discrete Fourier transform and various discrete cosine transforms.

For 20 constants, H_{cub} produces solutions with up to 17% fewer operations than RAG-n, and 25% fewer operations than BHM, where RAG-n is not applicable.



(a) $b = 12$ (b) $b = 16$ (c) $b = 19$ (d) $b = 22$ (e) $b = 28$ (f) $b = 32$

FIG. 12. Average number of $\mathcal{A}$-operations (adds/subtracts) vs. number of constants n for a fixed constant bitwidth b. Average is taken over 200 uniformly distributed random constant sets for $b \leq 22$, and over 50 sets for $b \geq 28$.

Fixed Constant Bitwidth. This experiment investigates the effect of changing n, the number of constants in the target set, for different fixed bitwidths $b = 12$, 16, 19, 22, 28, and 32. Figure 12 shows plots of the average number of operations (y axis) versus n (x axis) for 200 uniformly drawn random target sets of size n for $b \leq 22$, and 50 sets for $b \geq 28$.

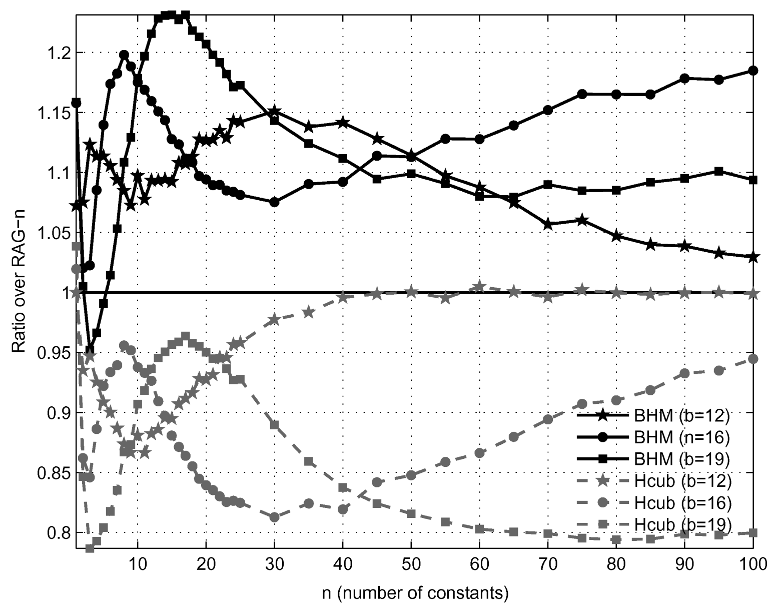


FIG. 13. Ratio of average number of $\mathcal{A}$-operations in the solution produced by H_{cub} and BHM over the average for RAG-n vs. n for a fixed constant bitwidth b. The average is taken over 100 uniformly drawn random target sets.

Again, in all cases H_{cub} outperforms all other algorithms. For 12 bits, both RAG-n and H_{cub} quickly converge to the optimal lower bound of n specified in Theorem 4.4. For 16 bits, more constants are needed to converge, and starting with 19 bits, we no longer see this behavior within the considered range of n. Note that this lower bound holds for n distinct odd constants (i.e., after right-shifting), however, on average the randomly drawn constant set contains slightly less than n unique odd constants after right-shifting.

For 16 bits and more, H_{cub} performs clearly the best, improving up to 20% over RAG-n for 16 and 19 bits. Beyond that, RAG-n is not applicable anymore, and the gap between H_{cub} and the next-best algorithm (i.e., BHM) widens. For example, at 28 bits, H_{cub} requires up to 26% less operations than BHM.

Comparison With RAG-n. To evaluate the performance improvement relative to BHM and RAG-n, we generated 100 randomly distributed constant sets, computed their MCM decompositions using RAG-n, BHM, and H_{cub}, and then plotted the ratio of average $\mathcal{A}$-operation counts of the solutions produced by H_{cub} and BHM over those produced by RAG-n versus n (see Figure 13) and versus b (see Figure 14). In the latter case, an optimal SCM decomposition was used for $n = 1$.

Figure 13 fixes the bitwidth $b = 12, 16, 19$, and plots the ratio versus n, the number of constants. The largest improvements of H_{cub} are observed for $b = 19$, with up to 20% lower operation counts than RAG-n at $n = 80$. BHM performs consistently worse than RAG-n.

Figure 14 fixes $n = 1, 2, 10, 20$, and plots the ratio versus b. Since for $n = 1$, an optimal decomposition is used, the ratio for $n = 1$ is always greater than 1. Otherwise, the average improvement over RAG-n tends to increase with b, but the ratio does not increase monotonically.

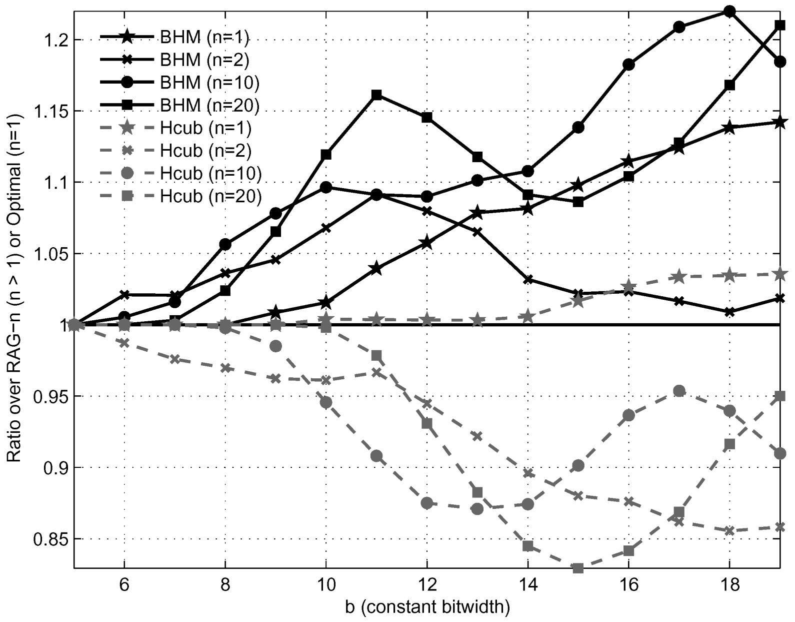


FIG. 14. Ratio of average number of $\mathcal{A}$-operations in the solution produced by H_{cub} and BHM over the average for RAG-n vs. b for a fixed constant set size n. The average is taken over 100 uniformly drawn random target sets.

*Effect of Distance-*3 *Tests.* The runtime of H_{cub} can be reduced by removing exact distance-3 tests and using only the estimator based on the distance-2 topology (i.e., Case 1 of Table IV). As discussed earlier, this reduces the asymptotic runtime to $O(n^3b^5)$.

Figure 15 shows the ratio of average $\mathcal{A}$-operation counts of the solutions produced by our algorithm with distance-2 tests only over our original variant with distance-2 and distance-3 tests. The averages were computed from 100 uniformly drawn random target sets.

For a single constant ($n = 1$), H_{cub} with distance-2 produces solutions with 6% to 15% higher operation counts. For $b \leq 19$ the largest difference occurs at $n = 1$ and eventually decreases to less than 3% for larger constant sets. For $b \geq 22$, however, the ratio initially rises and drops much more slowly. Since the average complexity of constants goes up with b, the importance of more precise distance tests should also increase, which is confirmed by the plot. Further, the plot shows that for increasing n, the difference between both tests eventually vanishes. The reason is that, intuitively, as the number of constants is increased, the precise distance value to a single target becomes less important, since the main objective is to optimize jointly for all targets.

Runtime. In Table VIII, we give a few average runtime examples of H_{cub} for one target set of varying size and bitwidth on a 3.4 GHz Pentium 4 EM64T Xeon workstation.

The runtimes are averages of 100 experiments, where each experiment was performed with a different random constant set. The runtimes show that the scope of parameters that should be sufficient for most applications is handled efficiently (by

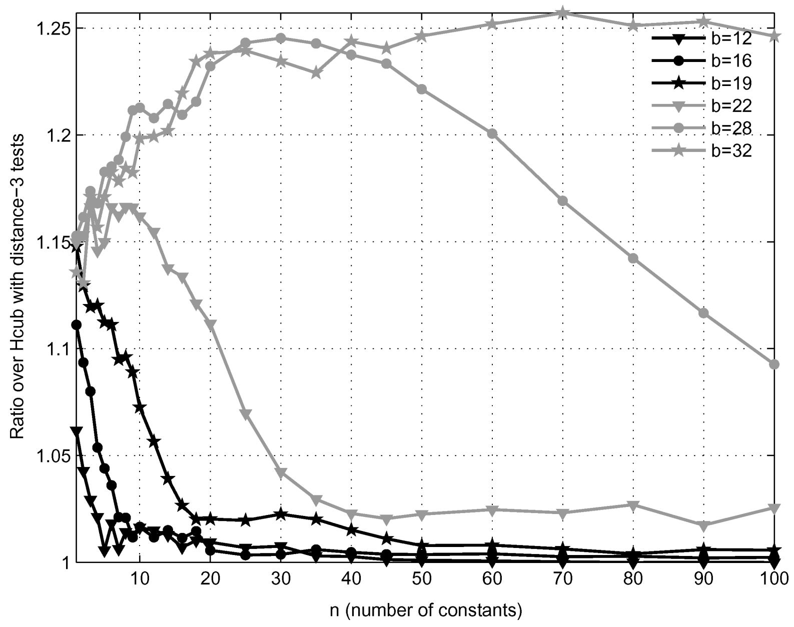


FIG. 15. Ratio of average number of $\mathcal{A}$-operations in the solution produced by H_{cub} with distance-2 tests only over the average for H_{cub} with distance-3 and distance-2 tests vs. n for several fixed constant bitwidths b. The averages are taken over 100 uniformly drawn random target sets.

TABLE VIII. AVERAGE RUNTIMES FOR H_{cub} IN SECONDS

b	$n = 1$	$n = 5$	$n = 10$	$n = 20$	$n = 50$	$n = 100$
12	.002 (.0004)	.003 (.002)	.007 (.006)	.012 (.012)	.020 (.020)	.063 (.063)
16	.002 (.001)	.023 (.013)	.060 (.038)	.16 (.11)	.64 (.60)	1.1 (1.0)
19	.007 (.002)	.092 (.035)	.29 (.14)	.80 (.50)	3.6 (2.9)	13 (13)
22	.018 (.004)	.28 (.071)	1.6 (.31)	3.9 (1.4)	16 (10)	60 (45)
28	.081 (.010)	1.9 (.20)	7.0 (1.0)	32 (6.0)	540 (60)	1800 (360)
32	.20 (.018)	4.2 (.35)	20 (1.8)	95 (11)	910 (150)	8500 (1040)

Average runtime for H_{cub} with distance-2 tests only is given in parenthesis. The averages are taken over 100 experiments with random constant sets. Values less than .01 are rounded to a single decimal digit.

as little as 8,500 seconds, i.e., about 2.5 hours for 100 constants of bitwidth 32 would be acceptable within a specialized hardware design for a digital filter).

Summary. In all performed experiments H_{cub} outperforms all other algorithms in terms of the number of $\mathcal{A}$-operations of the produced solutions. We achieve up to 20% improvement over RAG-n, the previously best available algorithm, while not being limited to 19-bit constants. The improvement comes at the expense of an increased runtime.

Although not shown in this article, we measured the standard deviation of the number of add/subtract operations for different algorithms on uniformly drawn random constant sets. Interestingly, the H_{cub} algorithm had the smallest standard deviation, followed by RAG-n and BHM, and CSD had the highest standard deviation.

7. *Conclusions*

The main contribution of this article is a new MCM algorithm that achieves significantly better results than previous methods, as we demonstrated for the cases most relevant in practice: bitwidth $b \leq 32$ and $n \leq 100$ constants. However, asymptotically, the new algorithm produces solutions with no known better complexity than $O(nb)$ add/subtract operations, just like CSD and all other algorithms.

The $\mathcal{A}$-distance computation and estimation framework developed in this article should be useful for further research in this area. One direction could be to improve the heuristic, which currently combines $\mathcal{A}$-distances in a trivial way. Another direction would be to use our framework to optimize MCM blocks with respect to other criteria, such as critical path, or to also minimize for the number of shifts required.

The big question that remains unanswered is the actual asymptotic worst-case cost of SCM and MCM decompositions. However, the precise bounds remain unknown, even for the simpler problem of addition chains.

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